ON THE LINEAR DILATATION OF QUASICONFORMAL MAPPINGS IN SPACE

By Marjatta Virsu

Let F_K^n denote the class of K-quasiconformal mappings from $\bar{R}^n$ onto itself preserving 0, 1, ∞, and let

$$\lambda_n(K) = \sup \{ |f(x)| : |x| = 1, f \in F_K^n \},$$

$$\mu_n(K) = \sup \{ H_f(x) : x \in \bar{R}^n, f \in F_K^n \},$$

where H_f is the linear dilatation of f.

For $n = 2$, a theorem of Teichmüller made possible explicit determination of λ_2 in terms of elliptic integrals, and at the same time, Lehto, Virtanen, and Väisälä [4] showed that $\lambda_2 = \mu_2$, using auxiliary conformal mappings and reflections to reproduce the distortion of the circle $|z| = 1$ by the Teichmüller extremal mapping on an infinite set of circles shrinking to $z = 0$.

To see whether similar results can be gained in the 3-dimensional case, we use in place of the Teichmüller mapping its rotation about the real axis and in place of auxiliary conformal mappings, the method of projection due to Gehring and Väisälä [3]. We then obtain the following estimates

$$\lambda_2((K/c_K)^{\frac{1}{3}}) \leq \mu_3(K) \leq \lambda_3(K); \quad c_K = 1 + (1 - K^{-4/3})^{\frac{1}{2}}, \tag{1}$$

$$\lambda_2(K) \leq \lambda_3(K). \tag{2}$$

In view of the expansion [4, Theorem 3]

$$\lambda_2(K) = \tfrac{1}{16}e^{\pi K} - \tfrac{1}{2} + \delta(K); \quad 0 < \delta(K) < 2e^{-\pi K}, \tag{3}$$

and the known upper bound e^{6K} for $\mu_3(K)$ [2, Lemma 8], (1) yields

$$\tfrac{1}{16} \exp (\pi(K/c_K)^{\frac{1}{3}}) - \tfrac{1}{2} < \mu_3(K) < \exp (6K), \tag{4}$$

and, since $\lambda_3(1) = \lambda_2(1) = 1$, $\mu_3(1) = 1$. For small K, $K > 1$, the lower bound in (4) is useless; by constructing a suitable mapping one gets the better estimate $K^2 \leq \mu_3(K)$. As $K \to \infty$, the lower bound in (4) grows as $\frac{1}{16} \exp (\pi \sqrt[3]{K/2})$.

1. Notation and definitions. We formulate the definitions for euclidean n-space R^n. We consider sets in $R^n \cup \{\infty\} = \bar{R}^n$, finite points are treated as vectors and designated by capital letters P, Q or small letters x, y. The coordinates for x are represented by $x_1, x_2, \cdots, x_n$ and the norm of x by $|x|$. The n-dimensional ball $|x - x_0| < r$ is denoted by $B^n(x_0, r)$ and its boundary sphere $|x - x_0| = r$ by $S^{n-1}(x_0, r)$. We use the abbreviations $B^n(0, r) = B^n(r)$,

Received October 30, 1969. This research was prepared at the University of Michigan under a Finnish State grant for young scientists.