THE DEGREE OF CONVERGENCE FOR ENTIRE FUNCTIONS

BY JOHN R. RICE

Dedicated to J. L. Walsh

1. Introduction. The main result of this paper is to characterize the set of entire functions of order $\rho > 0$ and type $0 < \tau < \infty$ in terms of their degree of convergence, on rather general sets. The preliminary results include extensions of some classical properties of entire functions. A generalization is indicated at the end for the approximation of a function with a finite number of singularities by sequences of rational functions. It is assumed that the reader is familiar with the terminology and results of [1] and [4].

One may obtain results about the degree of approximation on disks by the direct application of techniques from the theory of entire functions. Let $p_n(z)$ be the best approximation to $f(z)$ on a set D. Set

$$E_n = ||f(z) - p_n(z)||_D = \max_{z \varepsilon D} |f(z) - p_n(z)|.$$

Then for $D = \{z \mid |z| \leq r\}$ one sees that $f(x)$ is entire of order ρ and type τ if and only if

$$\lim_{n \to \infty} n^{1/\rho} \sqrt[n]{E_n} = \frac{1}{r} (e\rho\tau)^{1/\rho}.$$

These same techniques apply to more general sets, but the constants obtained are no longer sharp, see [3].

Bernstein [2] obtains similar results for approximation on $[-1, 1]$ by considering the expansion of $f(z)$ in terms of Tchebycheff polynomials of best least squares approximation with weight $(1 - x^2)^{\frac{1}{2}}$. He does not explicitly state a sharp result in the sense that the constants are determined, but he indicates the correct constants in his discussion [2, p. 114].

In this paper we establish the following

THEOREM 1. *Let C be a closed bounded point set whose complement is connected and regular and let $d_\infty(C)$ be the transfinite diameter of C. Given $f(z)$ defined on C then for the sequence of polynomials $p_n^*(z)$ of degree n of best approximation to $f(z)$ on C we have*

$$\lim_{n \to \infty} n^{1/\rho} ||f(z) - p_n^*(z)||_C^{1/n} = d_\infty(C)(e\rho\tau)^{1/\rho}$$

if and only if $f(z)$ is entire of order $\rho > 0$ and type $0 < \tau < \infty$.

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