NOTE ON FOURIER SERIES ON THE P-ADIC INTEGERS

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Introduction. The character group of the compact abelian group Δ_p of p-adic integers is isomorphic to the countable group $Z(p^\infty)$ and hence can be arranged into a sequence in several natural ways. We shall choose one such ordering, listing the characters in order as $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n, \cdots$. Given a function f in $L_1(\Delta_p)$ we shall then define the Fourier series of f to be the series $\sum_{m=1}^{\infty} \hat{f}(\gamma_m)\gamma_m$. In this paper we shall construct a function f in $L_1(\Delta_p)$ which has these two properties:

(a) The Fourier series of f diverges a.e.

(b) To each neighborhood V of the identity $\mathbf{0}$ of Δ_p there corresponds a trigonometric polynomial p_V on Δ_p such that $p_V(\mathbf{x}) = f(\mathbf{x})$ whenever $\mathbf{x}$ is not in V. Consequently the principle of localization fails rather dramatically for Fourier series of this type.

Vilenkin [3] has defined Fourier series for Haar-integrable functions on compact, 0-dimensional, abelian groups. Moreover, the principle of localization holds for such Fourier series [3; 15]. However, the manner in which Vilenkin arranges the characters into a sequence [3; 2] is different from the ordering defined below.

1. Definitions and notations.

1.1. Throughout this paper p will denote a fixed prime. K denotes the set of all complex numbers, Z the set of all integers, N the set of all natural numbers, and T the set $\{a \in K : |a| = 1\}$. The set Δ_p of p-adic integers is the Cartesian product $P_{i \in N} H_i$ where each H_i is the set $Z_p = \{0, 1, \cdots, p-1\}$. We denote p-adic integers by boldface letters such as $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc., or by sequences such as $(x_0, x_1, x_2, \cdots)$, $(y_0, y_1, y_2, \cdots)$, $(z_0, z_1, z_2, \cdots)$, etc. In fact, whenever a p-adic integer $\mathbf{x}$, $\mathbf{y}$, etc. appears in a discussion it is to be understood that $\mathbf{x} = (x_0, x_1, x_2, \cdots)$, $\mathbf{y} = (y_0, y_1, y_2, \cdots)$, etc. Of course, each x_i and y_i is an integer between 0 and $p - 1$.

The basic properties of the p-adic integers are set forth in [2, 10]. For each m in N we put $\Lambda_m = \{\mathbf{x} \in \Delta_p : x_0 = x_1 = \cdots = x_{m-1} = 0\}$. The family $\{\mathbf{x} + \Lambda_m : m \in N, \mathbf{x} \in \Delta_p\}$ is a base for a topology on Δ_p and, with this topology, Δ_p is a compact, Hausdorff, 0-dimensional, metric topological group. The sequence $\{\Lambda_m\}$ consists of compact open subgroups and forms a base at the identity $\mathbf{0} = (0, 0, 0, \cdots)$. λ will denote the normalized Haar measure of Δ_p and $L_1(\Delta_p)$ the group algebra of λ-integrable functions on Δ_p. We shall always write $\int_{\Delta_p} f(\mathbf{y})\, d\mathbf{y}$ in place of $\int_{\Delta_p} f(\mathbf{y})\, d\lambda(\mathbf{y})$. We can define a measure λ_i on the set H_i

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