REMARKS ON THE LOOP SPACE OF A FIBRATION

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Introduction. The fact that the existence of a cross section for a Hurewicz fibration $\mathcal{E} = (E, p, B)$ implies that the loop space of E splits in a natural way is well known and has appeared in many variants. On the other hand, the converse is certainly false for let $f:X \to Y$ be a map such that $\Omega(f):\Omega(X, x_0) \to \Omega(Y, y_0)$ is inessential but f is not, then, if $\mathcal{E}(Y)$ is the fibration of paths in Y based at y_0 the fiber space $f^{-1}(\mathcal{E}(Y))$ has no cross section but its loop space splits as a product (at least if the spaces involved are CW complexes). The purpose of this note is to point out that if we consider the loop space $\Omega(E, x_0)$ as a fibration over $\Omega(B, b_0)$ with fiber $\Omega(F, x_0)$, $F = p^{-1}(b_0)$, $x_0 \varepsilon F$, then its fiber homotopy type is essentially determined (See Corollary 2.1 for a precise statement) by the homotopy class of the map ϕ: $\Omega(B, b_0) \to F$ induced by a lifting function. (ϕ gives rise to the boundary homomorphism in the exact sequence of $\mathcal{E}$.) For instance, in the example above the class of ϕ is obviously zero. Our result contains the standard case for fibrations with cross sections but, in addition, it applies to situations where cross sections do not exist e.g., the generalized Whitney sum (§3).

Some remarks about notation and conventions. A fiber space means a triple $\mathcal{E} = (E, p, B)$, $p:E \to B$ continuous, which has the covering homotopy property (CHP). $\Omega(X, x_0)$ denotes the space of loops in X based at x_0, and x^* stands for the constant path at x all path spaces being given the C–0 topology. The word "map" will mean continuous map, and all spaces are assumed to be T_2. In so far as has been possible no restrictions have been imposed on the spaces involved and this, of course, has complicated some of the arguments e.g., in proving that a fiber map is a fiber homotopy equivalence we cannot appeal to the fact that it is a homotopy equivalence on fibers because the base space is not assumed to have any "nice" local properties.

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1. Fibrations associated to a map. Associated to a map $f : (X, x_0) \to (Y, y_0)$ between spaces with base points we consider the three fibrations below

$$\mathcal{E}(f) = (E(f), p_f, Y) \tag{1}$$

$$E(f) = \{(x, w) \mid x \varepsilon X, w:I \to Y, f(x) = w(0)\}$$

$$P_f(x, w) = w(1)$$

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