EXTENSIONS OF CHEVALLEY ALGEBRAS

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1. Introduction. Chevalley [1] has shown how to obtain a canonical basis for every finite-dimensional simple Lie algebra L over the complex numbers. The constants of structure with respect to this basis are integers, and this enables one to replace the complex scalars by members of an arbitrary field. In an earlier paper [2], we studied the ideal structure of the algebras L_R which arise if, instead of arbitrary fields, commutative rings R with identity are used to furnish the new scalars. Such algebras we called Chevalley algebras. In this paper we are concerned with families of extension algebras of each Chevalley algebra, and we determine the structure of the ideals in these extensions.

Let L be a simple Lie algebra of finite dimension over the complex field, H an n-dimensional Cartan subalgebra, S the ordered set of non-zero roots determined by H, and T the set of simple roots. If r and s belong to S, then let $p_{rs} = 0$ if $r + s$ is not a root; and in case $r + s$ is a root, let p_{rs} be the greatest integer i such that $s - ir$ is a root. Let $n_{rs} = 0 = p_{rs}$ in case $r + s$ is not a root; and let $n_{rs} = p_{rs} + 1$ if $r + s$ is a root. Let

$$B = \{e_r \mid r \in S\} \cup \{h_1, h_2, \cdots, h_n\}$$

be a Chevalley basis of L, that is, a basis satisfying

(1) $e_r e_{-r} = h_r$, where h_r is a certain (see [4; 2]) integral linear combination of the h_i,

(2) $h_i h_j = 0$ for all i and j, $1 \leq i, j \leq n$.

(3) $e_r e_s = \pm n_{rs} e_{r+s}$ if $r + s \neq 0$,

(4) $h_r e_s = c(s, r) e_s$, where $c(s, r) = 2(s, r)/(r, r)$ is the Cartan integer of s and r.

Let L_Z be the free abelian group on B. We denote by L_R the R-module $R \otimes_Z L_Z$, where R is a commutative ring with identity in which 2 and 3 are neither zero nor zero divisors. Then L_R is a free R-module generated by B. Moreover, it has the structure of a Lie algebra over R if we multiply in the natural way and interpret all integers in R. If $b \in B$, we identify b with $1_R \otimes b$, and write rb in place of $r \otimes b$. In [2] is found a determination of the ideals of L_R which the reader may wish to consult before proceeding, in view of the fact that parts of some arguments in [2] carry over verbatim to the context of the present paper and so are not repeated here.

Now let $\{h_i'\}_{i=1}^n$ be the basis for H over the complex field which is dual to the system T of simple roots. We have $h_i = \sum_{j=1}^n c(r_j, r_i) h_j'$, so that letting H_Z and H_Z' represent the $\mathbf{Z}$-modules generated by $\{h_1, h_2, \cdots, h_n\}$ and

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