# RADIAL ENGULFING IN CODIMENSION THREE

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**1. Introduction.** Let $M^n$ be a piecewise linear manifold without boundary, $U$ an open subset of $M^n$, $P$ a finite polyhedron in $M$, $Q$ a subpolyhedron of $P$ lying in $U$. Let $\dim Q \leq n - 3$, $\dim (P - Q) = r$.

Bing has proved the following general radial engulfing theorem [1].

Theorem A. (Bing) *Suppose* $r \leq n - 4$ *and* $\{A_\alpha\}$ *is a collection of sets such that finite r-complexes can be pulled into U along* $\{A_\alpha\}$. *Then for each* $\epsilon > 0$, *there is an engulfing isotopy* $H : M^n \times [0, 1] \to M^n$ *such that* $H_0 = id.$, $H_t = id.$ *on* $Q$, $P \subset H_1(U)$, *and for each* $x \in M^n$ *there are* $r + 1$ *elements of* $\{A_\alpha\}$ *such that the track* $H(x \times [0, 1])$ *lies in the* $\epsilon$*-neighborhood of the sum of these* $r + 1$ *elements.* (See [1] for definitions.)

Bing asks whether Theorem A holds when $r = n - 3$. We show that if one is content with a slightly larger bound on the orbits $H(x \times [0, 1])$, the answer is yes.

**2. Radial engulfing in codimension three.** For any subset $A \subset M^n$, let $N_\epsilon(A)$ denote the $\epsilon$-neighborhood of $A$ in $M^n$. In addition, we define the *double* $\epsilon$*-neighborhood* of an element $A_\alpha$ of the collection $\{A_\alpha\}$ as follows:

$$N^2_\epsilon(A_\alpha) = \{x \in M^n: x \text{ lies in } N_\epsilon(A_\beta) \text{ for some } A_\beta \text{ which intersects } N_\epsilon(A_\alpha)\}.$$

In generalizing Theorem A to the case $r = n - 3$, it is necessary to employ Zeeman's piping lemma. Piping apparently necessitates the use of the double $\epsilon$-neighborhood; certainly the theorem would be more pleasing if this concept could be avoided.

Theorem 1. *Let* $\dim P = n - 3$. *Suppose* $\{A_\alpha\}$ *is a collection of sets such that finite* $(n - 3)$*-complexes can be pulled into* $U$ *along* $\{A_\alpha\}$. *Then for each* $\epsilon > 0$ *there is an engulfing isotopy* $h_t : M \to M$ *such that* $h_0 = id.$, $h_t = id.$ *on* $Q$, $P \subset h_1(U)$, *and for each* $x \in M$ *there are* $n - 1$ *elements of* $\{A_\alpha\}$ *such that the track* $\{h_t(x)\}$ *lies in the union of the* $\epsilon$*-neighborhoods of* $n - 2$ *of these elements and the double* $\epsilon$*-neighborhood of the remaining element.*

*Proof of Theorem* 1. Let $H^1 : P \times I \to M$ be a homotopy such that $H^1(p, 0) = p$, $H^1[(P \times 1) \cup (Q \times I)] \subset U$, and for each $p \in P$, $H^1(p \times I)$ lies in some $A_\alpha$.

Let $T$ be a cylindrical triangulation of $P \times I$ such that for each $\sigma \in T$,

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