SYMMETRIC FUNCTION EXPANSIONS OF POWER SUM PRODUCTS INTO MONOMIALS AND THEIR INVERSES

By John Riordan

1. Introduction. The expansion of power sum products into monomial symmetric functions is possibly the oldest and surely the simplest expansion in the theory of symmetric functions; it is the first expansion tabled in the recent book [2] by F. N. David, M. G. Kendall, and D. E. Barton; and P. A. MacMahon in [3] seems to regard it as almost trivial. Nevertheless, the coefficients appearing are the Frobenius compound characters, which arise in the representation of the symmetric group, and if the ordinary monomials are replaced by the "augmented monomial symmetric functions", as in [2], many associations with the E. T. Bell multivariable polynomials may be made. The exposure of the latter, in the expansion mentioned and its inverse, is the object of the present paper, which also applies the congruences for Bell polynomials given by L. Carlitz in [1] to both expansions, thus providing checks on the numerical tables, usually computed in sequence.

2. Preliminaries. If the variables of the symmetric functions are taken as $x_1, x_2, \cdots, x_n$, the p-th power sum, usually denoted by s_p, is defined by $x_1^p + x_2^p + \cdots + x_n^p$. The ordinary monomial symmetric function is defined by the sum $\sum x_1^{p_1}x_2^{p_2}\cdots x_k^{p_k}$ over all k-permutations of the n variables if $p_1, \cdots p_k$ are distinct, and over such permutations yielding distinct terms otherwise; the function is denoted by the partition $(p_1p_2 \cdots p_k)$ of $w = p_1 + \cdots + p_k$; w is the weight of the function. The augmented monomial symmetric function, denoted, following [2], by $[p_1p_2 \cdots p_k]$, is the same sum over all k-permutations whether or not the parts $p_1, \cdots p_k$ are distinct. Hence

$$[p^iq^jr^k \cdots] = i!j!k! \cdots (p^iq^jr^k \cdots).$$

The Bell polynomials, in the notation of my book [4], are defined by $Y_0 = 1$, and

$$Y_n(g_1, \cdots, g_n) = \sum \frac{n!}{k_1! \cdots k_n!}\left(\frac{g_1}{1!}\right)^{k_1} \cdots \left(\frac{g_n}{n!}\right)^{k_n}, \qquad n = 1, 2, \cdots$$

with $k_1 + 2k_2 + \cdots + nk_n = n$ and summation over all partitions of n. They have two main recurrences which may be combined into the following:

$$\begin{aligned} Y_{n+1}(g_1, \cdots, g_{n+1}) &= \sum_{k=0}^{n} \binom{n}{k} g_{k+1} Y_{n-k}(g_1, \cdots, g_{n-k}) \\ &= (g_1 + D) Y_n(g_1, \cdots, g_n) \end{aligned}$$

Received May 5, 1969.