A GENERALIZED CONVOLUTION FOR ARITHMETIC FUNCTIONS

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1. Introduction. A great many papers have been written about the set A of arithmetic functions, often with a binary operation defined between arithmetic functions in order to give A some algebraic structure. Among the most common of these binary operations are the familiar Dirichlet convolution

$$(f * g)(n) = \sum_{dd'=n} f(d)g(d'),$$

and the so-called unitary product (as defined, for example, by Cohen [1])

$$(f \times g)(n) = \sum_{\substack{dd'=n \\ (d,d')=1}} f(d)g(d').$$

In spite of the differences between these two binary operations, many results which are phrased in terms of one of the operations are valid without essential change for the other operation and, in fact, for a large class of binary operations. Indeed, the Dirichlet convolution and the unitary product are just particular cases of convolutions over a basic sequence, and many arithmetical results depend only on the common properties of these convolutions.

2. Definitions. By a *basic sequence* $\mathcal{B}$ we mean a set of pairs (a, b) of positive integers with the properties

(i) $(a, b) \in \mathcal{B}$ if and only if $(b, a) \in \mathcal{B}$;
(ii) $(a, bc) \in \mathcal{B}$ if and only if $(a, b) \in \mathcal{B}$ and $(a, c) \in \mathcal{B}$;
(iii) $(1, k) \in \mathcal{B}(k = 1, 2, \cdots)$.

As examples of basic sequences we have $\mathcal{L}$, the set of all pairs of positive integers, and $\mathfrak{M}$, the set of all pairs of relatively prime positive integers.

For two arithmetic functions f, g and a basic sequence $\mathcal{B}$, we define the *convolution of f and g over* $\mathcal{B}$ by

$$(f \circ_{\mathcal{B}} g)(n) = \sum_{\substack{dd'=n \\ (d,d') \in \mathcal{B}}} f(d)g(d'). \tag{2.1}$$

The Dirichlet convolution is then just a convolution over $\mathcal{L}$, and the unitary product a convolution over $\mathfrak{M}$. The properties of basic sequences were investigated in [2], and the generalized convolution was discussed in [3].

For any basic sequence $\mathcal{B}$, the identity function δ under $\circ_{\mathcal{B}}$ is given by $\delta(1) = 1$, $\delta(n) = 0$ if $n > 1$. An arithmetic function f will have an inverse with respect

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