PARTITIONED HERMITIAN MATRICES

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1. Introduction. Let H be a positive, semi-definite mn-square hermitian matrix, partitioned into m^2 n-square blocks H_{st}, $s, t = 1, \cdots, m$. We denote this by

$$H = [H_{st}],$$

and the fact that H is positive semi-definite or positive-definite by $H \geq 0$ or $H > 0$ respectively. In general, if H_1 and H_2 are hermitian, then $H_1 \geq H_2$ means that $H_1 - H_2 \geq 0$.

Let $f: M_n \to M_r$ be a mapping of the set of n-square matrices M_n into the set of r-square matrices M_r. In this note we examine instances of the following question: for which f is it the case that $H \geq 0$ implies

$$H_f = [f(H_{st})] \tag{1}$$

is also positive semi-definite? Moreover, if $H > 0$, is it possible to obtain a lower bound on the minimum eigenvalue of H_f? All affirmative results contained in this paper are for $r = 1$. These problems have been considered by a number of authors. In a forthcoming paper [5], this problem is discussed for $f(X)$ an associated matrix in the sense of Schur [13]. In the paper [5], we specialize one of the results to the case $f(X) = E_q(X)$, the q-th elementary symmetric function of the eigenvalues of X to obtain a generalization of a result of de Pillis [2]. Some interesting inequalities between H and H_f are developed by one of the present authors [6], [7], de Pillis [1], Everitt [3], and Thompson [12]. It is an old result due to Schur [10] that if $n = 1$ and $f(x) = x^p$, where p is a positive integer, then $H_f = (f(h_{st})) \geq 0$. Quite recently Loewner [4] extended this to cover the case for any real number at least 1. It is not difficult to show that if $n = 1$ and $f(x) = |x|^2$, then the matrix $H_f = (|h_{st}|^2) \geq 0$ [8]. It is not true, however, that if $H = (h_{st}) \geq 0$ then $|H| = (|h_{st}|) \geq 0$. We are indebted to R. C. Thompson for the following example:

$$H = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 1 \end{bmatrix}.$$

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