UNIQUENESS THEORY FOR CESARO SUMMABLE HAAR SERIES

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1. Introduction. Much progress has been made in abstract Fourier analysis by proving analogues of well known trigonometric trsults. The purpose of this research was to investigate the Haar series analogue of a theorem due to Marcel Riesz [4]. Specifically:

THEOREM 1. *Suppose the Haar series*

$$S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$$

with $a_k = o(k^{\frac{1}{2}})$ *is Cesaro summable to a function* $f(x)$ *which is integrable and finite valued over* $[0, 1]$. *Then* S *is the Haar Fourier series of* $f(x)$.

This theorem is a corollary to Theorem 2.

We define the Haar functions by setting $\chi_0(x) \equiv 1$, $\chi_1(x) = 1$ if $0 \leq x < \frac{1}{2}$ and $\chi_1(x) = -1$ if $\frac{1}{2} < x \leq 1$ with $\chi_1(\frac{1}{2}) = 0$. For any integer $n > 1$ we write it uniquely as $n = 2^m + k$ where $0 \leq k < 2^m$, and as in [5] we define the intervals

$$\begin{aligned} \Delta(1, n) &= (k/2^m, (k + \tfrac{1}{2})/2^m) \\ \Delta(2, n) &= ((k + \tfrac{1}{2})/2^m, (k + 1)/2^m). \end{aligned} \tag{1}$$

Then the nth Haar function is defined as

$$\chi_n(x) = \begin{cases} \sqrt{2^m} & \text{if } x \in \Delta(1, n) \\ -\sqrt{2^m} & \text{if } x \in \Delta(2, n) \\ -\sqrt{2^m}/2 & \text{if } x = k + 1/2^m \\ \sqrt{2^m}/2 & \text{if } x = k/2^m \\ 0 & \text{otherwise} \end{cases}$$

$\Delta^*(i, n)$ will represent the closure of $\Delta(i, n)$.

Given a Haar series

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k \chi_k(x) \tag{2}$$

we define its Cesaro sum to be

$$\sigma(S, x) = \lim_{n\to\infty} \sigma_n(S, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_k(x). \tag{3}$$

The series (2) is said to satisfy *condition G* if for every $x_0 \in [0, 1]$,

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