GROUP EXTENSIONS BY TOTALLY DISCONNECTED GROUPS

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For topological groups G and H, define an extension of G by H to be a pair (E, σ) consisting of a topological group E which contains G as a closed invariant subgroup, and a continuous open homomorphism σ of E onto H whose kernel coincides with G. Two extensions (E_i, σ_i), $i = 1, 2$, are said to be equivalent if there is an isomorphism $\alpha: E_1 \to E_2$ of topological groups with $\sigma_1 \cdot \alpha = \sigma_2$. Given a continuous homomorphism $\eta: H \to \text{Aut}(G)/\text{Int}(G)$, the set of all equivalence classes of extensions of G by H with character η is denoted by $\text{Ext}(G, H, \eta)$.

The main purpose of this paper is to study the structure of $\text{Ext}(G, H, \eta)$ when G is a compact connected group, and H is a compact totally disconnected group. As every compact topological group is an extension of the above described type, this investigation enables us to look at the structure of compact groups from the point of view of group extensions. $\text{Ext}(G, H, \eta)$ has been characterized by A. Shapiro [5] for compact analytic groups, and by G. Hochschild [1], [2] for more general analytic groups, who has adopted a local argument by passing to the Lie algebras to overcome topological obstacles in the description of $\text{Ext}(G, H, \eta)$. Since his method is not applicable to the totally disconnected H, we had to rely on an improved version (see Theorem 2.3) of an earlier result of the author [4].

Throughout this paper, the identity element of a group is denoted by 1 and the center of a group G by $Z(G)$. Also $\mathfrak{C}_0$ (resp. ${}_0\mathfrak{C}$) denotes the category of compact connected (resp. compact totally disconnected) groups.

1. Existence of extensions.

1.1. Let $G \in \mathfrak{C}_0$ and let $\text{Aut}(G)$ denote the group of automorphisms of G. Under the compact open topology, $\text{Aut}(G)$ is a topological group. Since G is compact, the subgroup $\text{Int}(G)$ of inner automorphisms of G is a compact invariant subgroup of $\text{Aut}(G)$. As a well known result of Iwasawa [3], the quotient group $\text{Aut}(G)/\text{Int}(G)$ is totally disconnected.

1.2. Let $(G, H) \in \mathfrak{C}_0 \times {}_0\mathfrak{C}$, and let (E, σ) be an extension of G by H. For $a \in E$, the automorphism $x \to axa^{-1}$ of E is an inner automorphism of E whose restriction to G is an automorphism $r(a)$ of G. The map $a \to r(a)$ is a continuous homomorphism r of E into $\text{Aut}(G)$. Since $r(G) = \text{Int}(G)$, r induces a continuous homomorphism $\sigma^0: H(\cong E/G) \to \text{Aut}(G)/\text{Int}(G)$, which we call the *character*

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