EXTENDING HOMEOMORPHISMS BETWEEN APPROXIMATING POLYHEDRA

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1. Introduction. This is the last in a series of three papers investigating the relation between two locally tame approximations to a topological embedding of a polyhedron in a 3-manifold. In the first two papers [3], [4] we concentrate our attention on polyhedra with no local cut points. Here we consider arbitrary polyhedra. Our chief result is the following:

Theorem 3.2. *Suppose M is a 3-manifold with boundary, K is a polyhedron, K_a is a subpolyhedron of K, and f is a homeomorphism of K into M such that* $f^{-1}(\mathrm{Bd}\, M) = K_a$.

There is a positive, continuous function ν on K such that if f_0 and f_1 are homeomorphisms of K onto locally tame sets in M for which $f_e^{-1}(\mathrm{Bd}\, M) = K_a (e = 0, 1)$ and $\rho(f(x), f_e(x)) < \nu(x)$ $(e = 0, 1, x \in K)$, then there are neighborhoods N_0 of $f_0(K)$ and N_1 of $f_1(K)$ in M, and there is a homeomorphism h of N_0 onto N_1 such that $hf_0 = f_1$ and $h(N_0 \cap \mathrm{Bd}\, M) = N_1 \cap \mathrm{Bd}\, M$.

We also obtain a *pwl* version of Theorem 3.2.

Our notation conventions are discussed in [3], [4].

2. Homeomorphisms of relative regular neighborhoods of cones. We omit proofs of the first two lemmas here.

Lemma 2.1. *Suppose D is a disk, A is an arc whose intersection with D is a point $p \in \mathrm{Bd}\, A \cap \mathrm{Int}\, D$, M is 3-manifold with boundary, and f is a homeomorphism of $D \cup A$ into* Int M.

There is a $\delta > 0$ such that if f_0 and f_1 are homeomorphisms of $D \cup A$ into Int M *with $d(f, f_e) < \delta$ $(e = 0, 1)$ and $f_0 |D = f_1| D$, then $f_0(A)$ and $f_1(A)$ abut on the same side of $f_0(D)$.*

Lemma 2.2. *Suppose K is a polyhedron, v is a point joinable to K, B_0 and B_1 are pwl 3-cells, and f_0 and f_1 are pwl homeomorphisms of $v * K$ into B_0 and B_1 such that $f_e^{-1}(\mathrm{Bd}\, B_e) = K$, and B_e collapses to $f_e(v * K)$ $(e = 0, 1)$.*

Then if h is a pwl homeomorphism of Bd B_0 *onto* Bd B_1 *so that $hf_0 |K = f_1| K$, there is an extension of h to a pwl homeomorphism H of B_0 onto B_1 such that $Hf_0 = f_1$.*

Lemma 2.3. *Suppose K is a polyhedron, v is a point joinable to K, L is a subpolyhedron of $v * K$ which contains a neighborhood of each non-degenerate component of K, B is a pwl 3-cell, and f is a homeomorphism of $v * K$ into* Int B.

Suppose $K = K(1) \cup K(2)$ where $K(1)$ is a non-degenerate component of K.

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