QUASI-ANALYTICITY FOR FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. Given $\alpha > 0$ and a sequence of positive numbers $\{M_n\}$ $n = 0, 1, \cdots$, let $C_\alpha(\{M_n\})$ be the set of functions f analytic in the sector

$$S_\alpha = \{z: |\arg z| < \alpha\pi/2\}$$

such that for every n

(i) $f^{(n)}(z)$ has a continuous extension to $\bar{S}_\alpha$; and
(ii) $|f^{(n)}(z)| \leq M_n$ in S_α .

The class $C_\alpha(\{M_n\})$ is said to be quasi-analytic if $f \in C_\alpha(\{M_n\})$ and $f^{(n)}(0) = 0$ for every n together imply that $f \equiv 0$. Korenbljum [4; 233] has characterized the quasi-analytic classes by the following theorem.

THEOREM 0. *Let* $T(r) = sup_n\ r^n M_n^{-1}$. *Then* $C_\alpha(\{M_n\})$ *is quasi-analytic if and only if*

$$\int_1^\infty r^{-(\alpha+2)/(\alpha+1)} \log T(r)\, dr \tag{1}$$

diverges.

The Denjoy–Carleman theorem is the degenerate case ($\alpha = 0$) and, by a slight change of wording, the proof of the above theorem also proves the Denjoy–Carleman theorem. When $\alpha = 1$, we have a theorem for the half plane and hence for the unit disc.

In this paper we show that Theorem 0 extends to functions of several variables $z = (z_1, \cdots z_N)$. We define classes of functions by means of a multi-indexed sequence $\{M_n\}$, $n = (n_1, \cdots, n_N)$, by requiring that the derivatives of the function vanish on a portion of the boundary, namely:

$$Z = \bigcup_i \{z = (z_1, \cdots, z_N): z_i = 0\}.$$

Two immediate consequences of this theorem are a Denjoy–Carleman theorem for functions of several variables and a related theorem for functions in the polydisc. As a consequence of this last theorem, we can prove similar results by considering bounds on the Taylor coefficients rather than on the derivatives.

Throughout this paper we shall be working in the space of N complex variables. So, unless specifically stated otherwise, j will run through the index set

Received March 7, 1969. This paper represents a portion of a doctoral dissertation written under the guidance of Professor Peter L. Duren at the University of Michigan while the author was an NSF Trainee.