# LIE STRUCTURES IN SIMPLE GRADED RINGS

By Richard Speers

Herstein and others have proved theorems about the Lie structure of a simple associative ring. This paper is an investigation of what results are obtainable if the ring is graded, and one uses an appropriate definition of simplicity.

A graded ring $R = \bigoplus_{i \geq 0} R_i$ is a *simple graded ring* (sgr) if $R_i R_j \neq (0)$ for all $i$, $j$ and the only homogeneous ideals of $R$ are irrelevant. (An irrelevant ideal of a graded ring is one of the form $\bigoplus_{i \geq n} R_i$.)

Proposition 1. *Let $\bigoplus_{i \geq 0} R_i$ be a* sgr. *If $a \in R_j$, $a \neq 0$, then for all $i$ and $k$ we have $R_i a R_k = R_{i+j+k}$.*

*Proof.* Since $T = (\bigoplus_{r \geq i} R_r) a (\bigoplus_{s \geq k} R_s)$ is a homogeneous ideal of $R$, there either exists $n$ such that $T = \bigoplus_{t \geq n} R_t$ or $T = (0)$. If $T = \bigoplus_{t \geq n} R_t$, then $R_i a R_k = R_n$, and so $R_i a R_k = R_{i+j+k}$. Suppose $T = (0)$, and let $S = \{a \in R_j : R_i a R_k = (0)\}$. $S$ is clearly an additive subgroup of $R$, and if $c \in R_0$, $a \in S$, then $R_i c a R_j \subseteq R_i R_0 a R_j = R_i a R_j = (0)$. Hence, $ca \in S$; that is, $S$ is a left $R_0$-module. Thus, $S + \bigoplus_{r \geq j+1} R_r$ is a non-zero homogeneous ideal of $R$, so there exists $n$ such that $S + \bigoplus_{r \geq j+1} R_r = \bigoplus_{t \geq n} R_t$. This implies that $S = R_j$. Hence, $(0) = R_i R_j R_k = R_{i+j} R_k$, contradicting the simplicity of $R$.

If $R$ is a sgr, then $a R_0 b = (0)$ with $a$ and $b$ homogeneous implies that $a = 0$ or $b = 0$. For, if $a \in R_j$, $b \in R_k$ we have $R_0 a R_0 b R_0 = (0)$, so $R_j b R_0 = (0)$, and so $R_{j+k} = (0)$, a contradiction.

Let $R = \bigoplus R_i$ be a graded ring. If $x \in R_a$, $y \in R_b$ the Lie product (Jordan product) of $x$ and $y$ is defined by $[x, y] = xy - (-1)^{ab} yx ((x, y) = xy + (-1)^{ab} yx)$. Requiring that the product by bi-additive extends the definition to all of $R$. For ease of notation we write $[x, y] = xy - (-1)^{xy} yx$ if $x$ and $y$ are homogeneous. $[x, y]$ satisfies a Jacobi identity [4; 6]

$$(-1)^{ac}[[a, b], c] + (-1)^{bc}[[c, a], b] + (-1)^{ab}[[b, c], a] = (0).$$

The following identities will also be used:

$$[a, bc] = [a, b]c + (-1)^{ab} b[a, c]$$

$$[ab, c] = a[b, c] + (-1)^{bc}[a, c]b.$$

If $R$ is a simple ring, its center $Z(R)$ is a field, a very useful fact. This is clearly not true in a sgr; for, let $R = k[X]$, where $k$ is a field and $R$ is provided

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