SOME IDENTITIES IN COMBINATORIAL ANALYSIS

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1. The identities in question are

$$\prod_{n=1}^{\infty}(1-x^{7n})(1-x^{7n-3})(1-x^{7n-4}) = A\sum_{n=0}^{\infty} x^{2n^2}\frac{x_1x_3\cdots x_{2n-1}}{x_2x_4x_6\cdots x_{4n}}, \tag{1.1}$$

$$\prod_{n=1}^{\infty}(1-x^{7n})(1-x^{7n-2})(1-x^{7n-5}) = A\sum_{n=0}^{\infty} x^{2n(n+1)}\frac{x_1x_3\cdots x_{2n-1}}{x_2x_4x_6\cdots x_{4n}}, \tag{1.2}$$

$$\prod_{n=1}^{\infty}(1-x^{7n})(1-x^{7n-1})(1-x^{7n-6}) = A\sum_{n=0}^{\infty} x^{2n(n+1)}\frac{x_1x_3\cdots x_{2n+1}}{x_2x_4x_6\cdots x_{4n+2}}, \tag{1.3}$$

where

$$A = \prod_{k=1}^{\infty}(1-x^{2k})$$

and

$$x_n = 1 - x^n \qquad (n = 1, 2, 3, \cdots).$$

The identities were first proved by Rogers [3], [4] and rediscovered by Selberg [5]. Simpler proofs were given later by Dyson [2]. The present writer [1] has recently given a simple proof of the Rogers–Ramanujan identities that makes use of some properties of the "basic" Bessel function $I_n(t)$ defined by

$$\prod_{n=0}^{\infty}(1+x^n at)(1+x^n a^{-1}t) = \sum_{n=-\infty}^{\infty} a^n I_n(t). \tag{1.4}$$

The object of the present paper is to give a similar proof of (1.1), (1.2) and (1.3).

2. We shall require some easily proved properties of $I_n(t)$. In the first place it follows easily from (1.4) that

$$I_{2k}(t) = x^{k^2}\sum_{n=0}^{\infty}\frac{x^{n(n-1)}t^{2n}}{(x)_{n+k}(x)_{n-k}}, \tag{2.1}$$

$$I_{2k+1}(t) = x^{k(k+1)}\sum_{n=0}^{\infty}\frac{x^{n^2}t^{2n+1}}{(x)_{n+k+1}(x)_{n-k}}, \tag{2.2}$$

where as usual

$$(a)_0 = 1, \quad (a)_n = (1-a)(1-xa)\cdots(1-x^{n-1}a) \quad (n = 1, 2, 3, \cdots)$$

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