SOME RESULTS ON CONTRACTED IDEALS

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1. Introduction. If R is a subring of the commutative ring S, the first author in [7] considered conditions under which an ideal A of R is the contraction of an ideal of S, in both the local and global cases. The importance of such considerations in commutative ring theory is apparent, and we investigate this area further in this paper. Following [7], we say that R *has property* C *with respect to* S if each ideal of R is the contraction of an ideal of S. Gilmer in [7] was concerned primarily with determining when R has property C with respect to S in case R and S are integral domains, R is integrally closed, and S is integral over R. Originally we set out to determine whether $R[\{X_\lambda\}]$ has property C with respect to $S[\{X_\lambda\}]$, where $\{X_\lambda\}$ is an arbitrary set of indeterminates over S, when R has property C with respect to S. When R is a ring with identity and S is a unitary overring of R, we show in §2 that the answer to this question is affirmative if S is a faithfully flat R-module or if R is a direct summand of the R-module S. In §3 we show that the answer is also affirmative if R is either a Prüfer domain or a ring having a linearly ordered ideal system. In §4 we consider the three global conditions which Gilmer gives in [7] as being sufficient in order that R have property C with respect to S, where R is an integrally closed domain and S is a domain integral over R. We show that under any of Gilmer's three conditions, $R[\{X_\lambda\}]$ has property C with respect $S[\{X_\lambda\}]$.

The ramifications of considering property C between $R[\{X_\lambda\}]$ and $S[\{X_\lambda\}]$ go far beyond the question originally posed, however, and touch such unlikely subjects as Kronecker function rings, elementary divisor theory, and some congruential and approximation theorems in Krull domains. These ramifications arise from the following two results, each of which is easy to establish. If R is a ring with identity and S is a unitary overring of R, then 1) R has property C with respect to S if and only if each linear equation over R which is solvable over S is also solvable over R (Result 1), and 2) if each system of linear equations over R which is solvable over S is also solvable over R, then $R[\{X_\lambda\}]$ has property C with respect to $S[\{X_\lambda\}]$ (Theorem 1).

Throughout this paper we use "ring" to mean "commutative ring". If A is an ideal of a ring R which is a subring of the ring S, then A^e will denote the extension of A to S—that is, the ideal of S generated by A. What ring S is intended is usually clear from the context, and when it is not, we specify S explicitly. Under the same conditions on R and S, we use B^c to denote the contraction of an ideal B of S to R—that is, $B^c = B \cap R$.

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