ON THE SMOOTHNESS OF ISOMETRIES

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1. Main result. The object of this note is to prove the following:

THEOREM (*). *Let $x, y \in R^n$. Consider the unit balls $|x| < 1$, $|y| < 1$ to be Riemann manifolds M, N with the respective positive definite elements of arclength*

$$ds^2 = g_{ij}(x)\, dx^i\, dx^j, \tag{1.1}$$

$$ds^2 = h_{ij}(y)\, dy^i\, dy^j, \tag{1.2}$$

where $g_{ij} = g_{ji}$, $h_{ij} = h_{ji}$ are of class C^k. Let $y = f(x)$ be an isometric map of an open neighborhood of $x = 0 \in M$ onto a neighborhood of $y = 0 \in N$.

(i) *If $k = 0$, then $f(x)$ need not be of class C^1.*
(ii) *If $k = 0$ and $g_{ij}(x)$, $h_{ij}(y)$ have* (*uniform*) *degrees of continuity satisfying a Dini condition* (e.g., *if $g_{ij}(x)$ and $h_{ij}(y)$ are uniformly Hölder continuous*), *then $f(x)$ is of class C^1.*
(iii) *If $1 \leq k \leq \infty$, then $f(x)$ is of class C^{k+1}.*

Part (iii) is a strengthened form of a result of Myers and Steenrod [7] which states that if $k = 1$, then $y = f(x)$ is of class C^1. Their proof is not correct for it employs normal coordinates, but if $(g_{ij}(x))$ is only assumed to be of class C^1, then normal coordinates need not exist; cf. [1] or [5]. A proof of Part (iii) for $k = \infty$ is given in [6; 169–172]; this proof for $2 \leq k \leq \infty$ only yields the conclusion that $y = f(x)$ is of class C^{k-1}. For if $(g_{ij}(x))$ is of class C^k, $k > 1$, then the corresponding exponential maps are of class C^{k-1}, and need not be of class C^k; [2]. The arguments of Palais [8] (cf. [6; 169–172]) do not overcome the objections raised here.

Part (iii) implies the positive definite case of a theorem of [3] which states that if (g_{ij}), (h_{ij}) are symmetric, non-singular (possibly indefinite), and of class C^k, $1 \leq k \leq \infty$; and if $y = f(x)$ is of class C^1 and satisfies

$$h_{pm}(y)(\partial f^p/\partial x^i)(\partial f^m/\partial x^j) = g_{ij}(x),$$

then $f(x)$ is of class C^{k+1}. In view of this result, it would suffice to show that $f(x)$ is of class C^1 in the proof of Part (iii). But the proof below will not use [3].

In Part (i), (1.1) can be of class C^∞ (or reduce to the Euclidean metric $ds^2 = |dx|^2$), while the geodesics of (1.2) are not even differentiable functions of arclength. In particular, the example below proving Part (i) shows that

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