ON THE BOUNDARY CORRESPONDENCE UNDER CONFORMAL MAPPING

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Let $w = f(z)$ be a function mapping the disc $\{|z| < 1\}$ one-to-one and conformally onto a plane domain D. It is well known that for almost every $\theta(0 \leq \theta < 2\pi) f(z)$ has a finite radial limit $f(e^{i\theta})$ at $e^{i\theta}$, and consequently the image under $f(z)$ of the radius at $e^{i\theta}$ determines an (ideal) accessible boundary point $\mathfrak{a}_\theta$ of D whose complex coordinate $w(\mathfrak{a}_\theta) = f(e^{i\theta})$ is finite. The set of all such $\mathfrak{a}_\theta$ is denoted by $\mathfrak{A}$. We prove that if a subset $\mathfrak{E}$ of $\mathfrak{A}$ has a certain rather general geometrical property, then $\{\theta: \mathfrak{a}_\theta \in \mathfrak{E}\}$ is a set of measure zero, and we apply this result to prove that for almost every θ, D is in a sense of area large arbitrarily near $\mathfrak{a}_\theta$.

Fix $w_0 \in D$, and choose $r_0 > 0$ such that $|w_0 - w(\mathfrak{a})| > r_0$ for each $\mathfrak{a} \in \mathfrak{A}$. For each $\mathfrak{a} \in \mathfrak{A}$ and r satisfying $0 < r \leq r_0$ there exists a unique component $\gamma(\mathfrak{a}, r)$ of $D \cap \{|w - w(\mathfrak{a})| = r\}$ that separates $\mathfrak{a}$ from w_0 and can be joined to $\mathfrak{a}$ by an open Jordan arc lying in $D \cap \{|w - w(\mathfrak{a})| < r\}$. According to the lemma given below, there corresponds to each $\mathfrak{a} \in \mathfrak{A}$ an at most countable subset $N(\mathfrak{a})$ of the open interval $(0, r_0)$ having no accumulation point except possibly 0 such that for each $r \in (0, r_0)$ the set

$$U(\mathfrak{a}, r) = \bigcup_{r' \in (0,r) - N(\mathfrak{a})} \gamma(\mathfrak{a}, r') \tag{1}$$

is open. Note that $U(\mathfrak{a}, r)$ is contained in the component of $D \cap \{|w - w(\mathfrak{a})| < r\}$ through which $\mathfrak{a}$ is accessible. Let $L(\mathfrak{a}, r)$ and $A(\mathfrak{a}, r)$ denote the length of $\gamma(\mathfrak{a}, r)$ and the area of $U(\mathfrak{a}, r)$ respectively ($\mathfrak{a} \in \mathfrak{A}$, $0 < r < r_0$). We readily see that $A(\mathfrak{a}, r)$ is represented by the Lebesgue integral

$$A(\mathfrak{a}, r) = \int_0^r L(\mathfrak{a}, r)\, dr. \tag{2}$$

Finally, we say that a proposition $P(\mathfrak{a})$ concerning a point $\mathfrak{a} \in \mathfrak{A}$ holds almost everywhere (abbreviated a.e.) provided $\{\theta: P(\mathfrak{a}_\theta) \text{ is false}\}$ is a set of measure zero. Note that this definition is independent of f.

THEOREM 1.

$$\limsup_{r \to 0} \frac{A(\mathfrak{a}, r)}{\pi r^2} \geq \frac{1}{2} \quad \text{a.e.}$$

An immediate consequence of Theorem 1 is that

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