A CHARACTERIZATION OF THE EXTREME POINTS OF SOME CONVEX SETS OF ANALYTIC FUNCTIONS

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1. If A is a complex function algebra and K is a compact convex subset of the complex plane, then the set of functions in A taking values in K is convex. When K is the closed unit disc, then this set of functions is just the unit ball of A. An example of a theorem characterizing the extreme points of the unit ball of a function algebra is the following found in [1; 138].

Theorem 1. *f is an extreme point of the unit ball of H^∞ iff $|f(z)| \leq 1$ and $\int_{-\pi}^{\pi} \log (1 - |f(\theta)|)\, d\theta = -\infty$.*

In this paper we investigate the extreme points of the set of functions continuous in the closed disc D, analytic in the interior U with values in the compact convex set K. This set is denoted $ca(D; K)$. §3 characterizes the extreme points of $ca(D; K)$ where K is a convex polygon, §4, the extreme points where K is an ellipse. The aim throughout has been to prove theorems in a fairly special context and indicate later how they may be generalized.

2. This section is devoted to stating several classical theorems and to developing notation. The set of functions continuous in D and analytic in U is denoted $ca(D)$. The boundary values of a function $f \in ca(D)$ will sometimes be considered as functions on the unit circle, other times (such as when differentiability is involved) as periodic functions. $f(\theta)$ will be used to denote the boundary function. ∂K denotes the topological boundary of the set K. $d(w_1, w_2)$ denotes $|w_1 - w_2|$. If w is a point and E a set, $d(w, E) = \inf_{y \in E} |w - y|$.

$P_r(t) = \operatorname{Re} \dfrac{1 + re^{it}}{1 - re^{it}}$ is the Poisson kernel.

$Q_r(t) = \operatorname{Im} \dfrac{1 + re^{it}}{1 - re^{it}} = \dfrac{2r \sin t}{1 - 2r \cos t + r^2}$ is the conjugate Poisson kernel.

Theorem 2. [1; 78]. *If $u(\theta) \in L^1$ of the circle, then*

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta - t) P_r(t)\, dt$$

gives the harmonic extension of u to the disc.

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