# METRIC CONDITIONS FOR RATIONAL APPROXIMATION

By John Garnett

**1. Introduction.** Let $X$ be a compact subset of the complex plane, and let $A(X)$ be the algebra of continuous functions which are analytic on $X^0$, the interior of $X$. Denote by $R(X)$ the uniformly closed subalgebra of $A(X)$ generated by those functions analytic on a neighborhood of $X$. We regard each function as extended to $S^2$, the Riemann sphere. We seek metric conditions which imply that a function in $A(X)$ lies in $R(X)$. Now the boundary of $X$ decomposes into the *outer boundary*, which is the union of the boundaries of the complementary components, and the *inner boundary*, which is the relative complement of the outer boundary. Let $E$ denote the inner boundary of $X$. In light of the theorem of Vitushkin [8], [9] asserting that the functions in $A(X)$ analytic on the outer boundary of $X$ are uniformly dense in $A(X)$, we only seek conditions involving the inner boundary $E$. Given here are three conditions for approximation: one depending on $E$ and $X$, one on $E$ alone, and one on $E$ and the function $f$.

**2. A condition on $E$ and $X$.** Let $V_\delta = V_\delta(E) = \{z : \text{dist}(z,E) < \delta\}$, and let $m$ denote the area measure.

Theorem 2.1. *Assume $X$ is a compact set with inner boundary $E$, and*

$$\varliminf_{\delta\to 0} \frac{m(V_\delta \cap X)}{\delta^2} < \infty. \tag{2.1}$$

*Then $R(X) = A(X)$.*

*Proof.* Write

$$M = \varliminf_{\delta\to 0} \frac{m(V_\delta \cap X)}{\delta^2}. \tag{2.2}$$

Let $f \in A(X)$ and $\epsilon > 0$. Since $m(\bar{E}) = 0$, we have $R(\bar{E}) = A(\bar{E})$ by the theorem of Hartogs and Rosenthal [9], so that there is a function $g$ holomorphic in $\bar{E}$ with $|f(z) - g(z)| < \epsilon$ on $\bar{E}$. Choose $\delta$ such that

(i) $|f(z) - g(z)| < \epsilon$ on $V_{2\delta}$
(ii) $m(V_{2\delta} \cap X) < 5M\delta^2$
(iii) $g$ is analytic on $V_{2\delta}$ .

Received November 13, 1968. Partially supported by the Air Force Office of Scientific Research, Office of Aerospace Research, under Contract No. F44620-67-6-0029, and by NSF Grant GP-8622.