CHARACTERIZATION OF DIMENSION IN TERMS OF THE EXISTENCE OF A CONTINUUM

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1. Introduction. The results of this paper arose during a study of metric dependent dimension functions. A useful tool for this study, Theorem 1, which is proved in Part I, is a purely topological result which characterizes covering dimension being greater than or equal to n for compact Hausdorff spaces, in terms of the existence of a continuum. In Part II, we apply Theorem 1 to obtain lower bounds for the metric dimension of certain spaces. Our principal result in this regard is Theorem 2.

Theorem 1. *Let X be a compact Hausdorff space and $\mathfrak{C}_n = \{C_1, C_1'; C_2, C_2'; \cdots; C_n, C_n'\}$ be a collection of n pairs of closed subsets of X with C_i missing C_i' for each i, $1 \leq i \leq n$. Then $\mathfrak{C}_n$ is an n-defining system for X iff for every finite closed cover $\mathfrak{F}$ of X with small mesh relative to $\mathfrak{C}_n$, $\{x : x \in X$ and $\text{ord}_x \mathfrak{F} \geq n\}$ contains a continuum hitting all $2n$ elements of $\mathfrak{C}_n$.*

Theorem 2. *Let (X, ρ) be a compact metric space, $\mathfrak{C}_n = \{C_1, C_1'; \cdots; C_n, C_n'\}$ an n-defining system for X, $\{A_i\}$ a countable collection of closed subsets of X and m an integer with $-1 \leq m \leq n - 1$. Suppose moreover that*

a) $\dim A_i \leq n - 1$ *for all* i
b) $\dim (A_i \cap A_j) \leq m$ *for all* $i \neq j$
c) *No component of any* A_i *hits all* $2n$ *elements of* $\mathfrak{C}_n$.

Then $\mu \dim ((X - \bigcup_{i=1}^{\infty} A_i), \rho) \geq n - (m + 2)$.

The statement "$\mathfrak{F}$ is of small mesh relative to $\mathfrak{C}_n$" means that no $F \in \mathfrak{F}$ hits both C_i and C_i' for any i. "$\mathfrak{C}_n$ is an n-defining system for X" means that if B_i is a closed set separating C_i from $C_i' (i = 1, 2, \cdots, n)$, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. The existence of an n-defining system for a normal space X is equivalent to dim $X \geq n$. (This is the Eilenberg–Otto characterization of covering dimension. See [2] and [5]). The order of a point x of a set S relative to a collection $\mathfrak{F} = \{F_\alpha : \alpha \in A\}$ of subsets of S is denoted by $\text{ord}_x \mathfrak{F}$ and is defined by

$$\text{ord}_x \mathfrak{F} = \text{cardinal number of } \{\alpha : \alpha \in A \quad \text{and} \quad x \in F_\alpha\}.$$

By $\mu \dim (X, \rho)$, we mean the metric dimension of X with respect to the metric ρ and we understand a continuum to be a compact, connected set. For basic information on μ dim, as well as on other metric dependent dimension functions, see [7].

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