EXTENSION OF LINEAR FUNCTIONALS ON F-SPACES WITH BASES

By Joel H. Shapiro

1. Introduction. A linear topological space is said to have the Hahn-Banach Extension Property (HBEP) if every continuous linear functional on a closed subspace has a continuous linear extension to the whole space. Duren, Romberg, and Shields [4, §7] give an example, due to A. Shuchat, of a non-locally convex space with the HBEP; and ask if this can happen in a non-locally convex F-space. Here we show that the answer is negative for F-spaces with a basis. For this class of spaces, then, the HBEP and local convexity are equivalent. The proof is in §3, with the necessary background material occupying §2.

2. Background material. An *F-space* is a complete linear metric space over the real or complex field. If E is an F-space, there is a complete translation invariant metric d in E for which the functional $||x|| = d(x, 0)$ is an *F-norm*, that is:

(a) $||x|| \geq 0$ for all x in E, and $||x|| = 0$ iff $x = 0$,
(b) $||x + y|| \leq ||x|| + ||y||$,
(c) $||\alpha x|| \leq ||x||$ whenever $|\alpha| \leq 1$,
(d) $\lim_{n\to\infty} ||x/n|| = 0$ for each x in E,
(e) the metric $d(x, y) = ||x - y||$ is complete.

Conversely, if E is a real or complex linear space, and $||\cdot||$ is an F-norm on E, then $d(x, y) = ||x - y||$ defines a metric under which E becomes an F-space (see Kelley–Namioka [5; 52]). We say two F-norms on E are *equivalent* if they induce the same topology on E.

The interior mapping principle and the principle of uniform boundedness hold for F-spaces (see Dunford and Schwartz [3, Chapter II]).

From now on, E denotes an F-space whose topology is induced by an F-norm $||\cdot||$. E' is the (continuous) dual of E. A sequence $\{e_n\}_0^\infty$ in E is called a *basis* if to each $x \in E$ there corresponds a unique sequence $\{\xi_n(x)\}_0^\infty$ of scalars such that the series $\sum_{n=0}^\infty \xi_n(x)e_n$ converges in E to x. The coordinate functionals ξ_n are clearly linear, and are continuous (see Corollary to Proposition 1). A sequence $\{e_n\}$ in E is called *basic* if it is a basis for the closed subspace it spans. The following result is essentially proved by Arsove [1].

Proposition 1. *Suppose $\{e_k\}_0^\infty$ is a basis in E. Then*

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