RESIDUE CLASS CHARACTERS

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H. Hasse [1], [2] was the first to give a connected account of that portion of the theory of residue class characters having to do with primitivity, resolving moduli and conductors. In the application of these results to cyclotomic fields (Lepistö [4; 12–14]), there are (non-obvious) gaps between the results of Hasse, and what is necessary there. Prachar [5] gave a portion of a different development of the theory of characters, and then referred back to Hasse's development for the remaining results. In this paper, we use the method introduced by Prachar to prove the refined results needed for cyclotomic fields. It would also be possible to prove these results using the concrete definition of characters necessary for automatic computations (Spira [6]), but the theory becomes easy and elegant if we use a simplication of the characterization of characters given by Prachar [5]. We take here this characterization as the definition, and otherwise make use only of the elements of number theory. At the end, we show that the definitions of primitivity of Landau [3] and Prachar [5] coincide, and discuss the different notion of primitivity of Hasse [1], [2].

Definition 1. A complex function $f(n)$ defined on the integers is a character mod k, $(k \geq 1)$, if

a) $f(n)\begin{cases} =0 & \text{if} \quad (n, k) > 1, \\ \neq 0 & \text{if} \quad (n, k) = 1, \end{cases}$

b) $(m, k) = (n, k) = 1$ and $m \equiv n (\text{mod } k)$ imply $f(m) = f(n)$,

c) $(m, k) = (n, k) = 1$ imply $f(mn) = f(m)f(n)$.

It is easy to see that the restriction $(m, k) = (n, k) = 1$ could be dropped from b) and c), but for the proofs we wish to have a tight characterization.

Definition 2. Two characters χ_1 (mod k_1) and χ_2 (mod k_2) are *equivalent*, written $\chi_1 \approx \chi_2$ if $\chi_1(n) = \chi_2(n)$ for all n such that $(n, k_1) = (n, k_2) = 1$.

Definition 3. If, for a character χ_1 (mod k_1) and a natural number k_2 there is a character χ_2 (mod k_2) such that $\chi_1 \approx \chi_2$, we say χ_1 is *resolvable* by k_2, and that k_2 is a *resolving modulus* for χ_1, and that χ_2 is a *resolving character* of χ_1.

Lemma 1. $\chi_1 \approx \chi_1$. $\chi_1 \approx \chi_2$ *implies* $\chi_2 \approx \chi_1$.

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