# THE RADIUS OF THE ESSENTIAL SPECTRUM

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**Introduction.** Let $T$ be a closed, densely defined linear operator on a Banach space $X$. F. E. Browder [1] defined the essential spectrum of $T$, ess $(T)$, to be the set of $\lambda \varepsilon \sigma(T)$, the spectrum of $T$, such that at least one of the following conditions holds: (1) $R(\lambda - T)$, the range of $\lambda - T$, is not closed; (2) $\lambda$ is a limit point of $\sigma(T)$; (3) $\bigcup_{r \geq 1} N(\lambda - T)^r$ is infinite dimensional, where $N(A)$ denotes the null space of a linear operator $A$. Browder proved that $\lambda_0 \notin$ ess $(T)$ iff for some $\delta > 0$, $\lambda$ is in the resolvent set of $T$ for $0 < |\lambda - \lambda_0| < \delta$ and the Laurent expansion of $(\lambda - T)^{-1}$ around $\lambda_0$ has only a finite number of non-zero coefficients with negative indices.

In this paper we shall consider bounded $T$ and we shall obtain natural formulas, analogous to the usual spectral radius formula, for $r_e(T) = \sup \{|\lambda| : \lambda \varepsilon \text{ ess } (T)\}$. The basic tool we shall use is the "measure of noncompactness," a notion which was introduced by C. Kuratowski in 1930 [7].

1. We begin with some definitions. Let $X$ be a complete metric space and $A$ a bounded subset of $X$. Following Kuratowski [7], we define $\gamma(A)$, which we shall call the measure of noncompactness of $A$, to be inf $\{d > 0$: there exists a finite number of sets $S_1, \cdots, S_n$ such that diameter $(S_i) \leq d$ and $A = \bigcup_{i=1}^n S_i\}$. Though we shall use the measure of noncompactness only for linear maps, we should remark that it is a useful tool in considering nonlinear problems. For instance, G. Darbo [2] and the author [8], [10] have obtained fixed point theorems for nonlinear maps by utilizing the measure of noncompactness.

If $A$ and $X$ are as above, we define the ball measure of noncompactness of $A$ in $X$, $\tilde{\gamma}_X(A)$, to be inf $\{r > 0$: there exists a finite number of balls $V_1, \cdots, V_n$ with centers in $X$ and radii $r$ such that $A \subset \bigcup_{i=1}^n V_i\}$. The ball measure of noncompactness was introduced by Goldenstein, Gohberg, and Markus [3] and later studied by Goldenstein and Markus [5]. Apparently they were unaware of the work of Kuratowski and Darbo. The reader should note that our terminology differs from that in [5]: Goldenstein and Markus call the ball measure of noncompactness of $A$ in $X$ simply the measure of noncompactness and write $q_X(A)$ instead of $\tilde{\gamma}_X(A)$.

The reason for our terminology is simple: since a complete metric space is compact iff it is totally bounded, $\gamma(A) = 0$ iff $\tilde{\gamma}_X(A) = 0$ iff $A$ has compact closure.

If $X$ is a Banach space and $A$ and $B$ are bounded subsets of $X$, Darbo [2] has shown that $\gamma(A + B) \leq \gamma(A) + \gamma(B)$, where $A + B = \{a + b: a \varepsilon A, b \varepsilon B$ and $\gamma$ (convex hull of $A$) $= \gamma(A)$. In a similar way one can prove that

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