METRIC-DEPENDENT FUNCTION d_2 AND COVERING DIMENSION

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1. Introduction. In [5], K. Nagami and the author introduced the metric-dependent function d_2, defined for every metric space (X, ρ).

Definition. $d_2(X, \rho)$ is the smallest integer n (if such integer exists) such that if $C_1, C'_1; \cdots; C_{n+1}, C'_{n+1}$ are $n + 1$ pairs of closed sets with $\rho(C_i, C'_i) > 0$ for $i = 1, 2, \cdots, n + 1$, then there exist closed sets $B_1, \cdots, B_{n+1}$ with B_i separating C_i from C'_i in X and such that $\bigcap_{i=1}^{n+1} B_i = \emptyset$. If no such integer n exists, then $d_2(X, \rho) = \infty$.

It is natural to think of d_2 as "Eilenberg–Otto positive distance dimension", for the following reason: If the requirement "$\rho(C_i, C'_i) > 0$" is replaced by "$C_i \cap C'_i = \emptyset$", one obtains the Eilenberg–Otto characterization of covering dimension (applicable even if the space is merely normal). (See [1], [2], and [4].) This function d_2 is closely related to metric dimension, denoted μ dim, where μ dim (X, ρ) is the smallest integer n such that for every $\epsilon > 0$ there exists an open cover of X of mesh $< \epsilon$ and order $\leq n + 1$. In [6] it is shown that $d_2(X, \rho) \leq \mu$ dim (X, ρ), and for every integer $n \geq 2$ an example X_n is constructed such that $d_2(X_n, \rho) = [n/2] < \mu$ dim $(X, \rho) = n$. Now Katetov [3] has shown that 2μ dim $(X, \rho) \geq$ dim X (covering dimension). The purpose of the present paper is to prove this same result for d_2.

Theorem. *For every non-vacuous metric space (X, ρ), $2d_2(X, \rho) \geq$ dim X.*

2. Intuitive guide to the proof. Let (X, ρ) be a fixed non-vacuous metric space, set $d_2(X, \rho) = k$ (there is nothing to prove if $d_2(X, \rho) = \infty$), and let $C_1, C'_1; \cdots; C_{2k+1}, C'_{2k+1}$ be $2k + 1$ pairs of closed sets with $C_i \cap C'_i = \emptyset$. The aim is to prove dim $X \leq 2k$ by finding closed sets $B_1, \cdots, B_{2k+1}$ such that B_i separates C_i from C'_i, and $\bigcap_{i=1}^{2k+1} B_i = \emptyset$—the Eilenberg–Otto characterization. To apply our hypothesis that $d_2(X, \rho) = k$ we need sets C^*_i, C'^*_i at positive distance, and the function α (§3) leads to a breakdown $C_i = \bigcup_{j=1}^{\infty} C_{ij}$ and $C'_i = \bigcup_{j=1}^{\infty} C'_{ij}$ such that $\rho(C_{ij}, C'_{ij}) > 0$ for every i and j. It would be appreciably easier to prove that $2d_2(X, \rho) + 1 \geq$ dim X, because in that case we would have $2k + 2$ $(= 2(k + 1))$ pairs C_i, C'_i. In the actual case a pair of level surfaces of the function α serves as the $(2k + 2)$-th pair, in some applications of the hypothesis.

3. The function α, sets D_j, M_j and F_j. We want to have a real function $\alpha: X \to (0, \infty)$, such that for every $\epsilon > 0$, in the space $X_\epsilon = \{x: \alpha(x) \geq \epsilon\}$

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