ELEMENTARY EXTENSIONS WITHIN REDUCED POWERS

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If $\mathfrak{A}$ is a relational system and D a filter on an index set I then we can form the reduced power $\mathfrak{A}^I/D$ (see [1]) and with the natural identification, $\mathfrak{A} \subseteq \mathfrak{A}^I/D$. The purpose of this paper is to discuss elementary extensions $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}^I/D$. (We restrict our attention to countable languages.) If D is an ultrafilter, a basic result due to Łoś [4] and proved in [1] states that for any formula $\varphi(x_1, \cdots, x_n)$ and any $b_1/D, \cdots, b_n/D$ in $\mathfrak{A}^I/D$, $\mathfrak{A}^I/D \models \varphi(b_1/D, \cdots, b_n/D)$ if and only if $\{i \mid \mathfrak{A} \models \varphi(b_1(i), \cdots, b_n(i))\} \in D$. If, however, D is just a filter on I, then in general $\mathfrak{A}$ and $\mathfrak{A}^I/D$ have different theories, and Łoś' theorem does not hold. We call subsystems $\mathfrak{B}$ of $\mathfrak{A}^I/D$ which do satisfy Łoś' theorem Łoś subsystems. These, of course, are elementary extensions of $\mathfrak{A}$, as is $\mathfrak{A}^I/D$ if D is an ultrafilter.

Since $\mathfrak{A}$ and $\mathfrak{A}^I/D$ have in general different theories, there are two embedding problems. One problem is to determine when $\mathfrak{A}^I/D$ is universal. Some results in this direction have been obtained by Galvin [2]. Another problem is to determine when systems which are elementarily equivalent to $\mathfrak{A}$ and have power $\leq$ that of $\mathfrak{A}^I/D$ can be embedded in $\mathfrak{A}^I/D$. In §3 below we show that if D is an (ω, α)-regular filter on a set I, then every elementary extension of $\mathfrak{A}$ of power $\leq \alpha$ can be embedded in $\mathfrak{A}^I/D$ as a Łoś subsystem. If, in addition, we assume that $\alpha = \omega$ and that $\mathfrak{A}$ has cardinality ω, then every elementary extension of $\mathfrak{A}$ of power $\leq \omega_1$ can be embedded in $\mathfrak{A}^I/D$ as a Łoś subsystem. The method of proof resembles methods used by Keisler in [3] and extends techniques of [5]. As a corollary we get that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ give rise to isomorphic Łoś systems.

In §1 properties of Łoś systems are discussed and a characterization due to M. Morley given. Under the added assumption of $\mathfrak{A}$ having definable Skolem functions, yet another characterization of Łoś systems is given and a particular subclass of Łoś systems is defined.

In §2 the existence of non-trivial Łoś systems is discussed ($\mathfrak{A}$ is always Łoś), and it is shown that a sufficient condition for this to happen is that D be an (ω, α)-regular filter, where α is the cardinality of the universe of $\mathfrak{A}$.

Some of these results, including Theorem 3.1, were announced in [6]. It was stated there that Theorem 3.1 held for $c(\mathfrak{C}) \leq \alpha^+$ and this was later corrected to read $c(\mathfrak{C}) \leq \alpha$. The first statement of the theorem remains an open question for $\alpha > \omega$, even when D is an ultrafilter.

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