OHM'S PROPERTY (n) FOR FIELD EXTENSIONS

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1. Introduction. For n a positive integer and D an integral domain, Ohm has defined in [2] the statement "D has property (n)". More generally, he has defined what it means for a ring R to "have property (n)" with respect to a unitary overring S of R. The definition is as follows: For each $\xi \in S$, and for $1 \leq i \leq n - 1$, there exist a_i , $b_i \in R$ such that $\xi^i = a_i\xi^n + b_i$. It had been conjectured that an integral domain having property (n) for each positive integer n was a Prüfer domain. (A Prüfer domain is an integral domain D such that each finitely generated ideal of D is invertible.) To disprove this, Ohm showed that it was sufficient to find a field K and a simple algebraic extension field $K(t)$ of K such that K has property (n) with respect to $K(t)$ for each positive integer n. Gilmer in [1] considered the following question: Suppose K is a field, $K(t)$ a simple algebraic extension field of K, and $[K(t):K] = m$. For n a positive integer, give necessary and sufficient conditions on K and $K(t)$ in order that K have property (n) with respect to $K(t)$. Ohm in [2] had solved the problem for arbitrary m when $n = 2$. Gilmer solved the problem when $m = 2$ for the cases $n = 3$ and $n = 5$.

In §2 we give a complete solution to Gilmer's question. Namely, we prove that if $m \geq 3$, K does not have property (n) with respect to $K(t)$ for any n. Further, we prove that for $[K(t):K] = 2$ and an odd integer n, K has property (n) with respect to $K(t)$ if and only if each n-th root of unity that is in $K(t)$ belongs to K. Since for $[K(t):K] = 2$ and positive integers r and s, K has property (rs) with respect to $K(t)$ if and only if K has properties (r) and (s) with respect to $K(t)$, we therefore obtain, using Ohm's result when $n = 2$, necessary and sufficient conditions in order that K have property (n) with respect to $K(t)$ for any n.

The most difficult part of the paper is §3 where we answer a conjecture of Gilmer's concerning property (n) in the affirmative if n is odd, and in the negative if n is even.[3]

Our notation and terminology are essentially that of Zariski–Samuel [4].

2. Characterization of property (n) for simple field extensions. In this section we deduce a chain of results which enables us to give necessary and sufficient conditions on K and $K(t)$ in order that K have property (n) with respect to $K(t)$.

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