THE DISTRIBUTION OF k-th POWER NON-RESIDUES

By JAMES H. JORDAN

1. Introduction. For a prime $p \equiv 1 \pmod{k}$, $k \geq 2$, the reduced residue system, S, modulo p, is a multiplicative group and has a proper multiplicative subgroup, R_k, consisting of the kth power residues modulo p. The $k - 1$ cosets formed with respect to R_k, say $N^1, N^2, \cdots, N^{k-1}$, are called classes of kth power non-residues modulo p. If $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ are the smallest positive integers in $N^1, N^2, \cdots, N^{k-1}$ respectively then a bound $f(k, p)$ is sought so that $\alpha_i \leq f(k, p)$ for $i = 1, 2, \cdots, k - 1$. The smallest function is desirable.

In this paper we establish that for any $\epsilon > 0$, $p^{(1-d)/4+\epsilon}$ is an upper bound when $p > P_\epsilon$ and where d, which depends on k, is a positive constant expressed as the solution of an equation involving sums of multiple integrals with variable limits of integration. The result is an improvement over the previous known results.

2. History. The problem was first investigated by Vinogradov [9] who proved

THEOREM A. *Let $k \mid p - 1$ and N^i, $i = 1, 2, \cdots, k - 1$ be the class of non-resides modulo p and $N^i(x)$ denote the number of positive integers in N^i that are $\leq x$. Then, $N^i(x) = x/k + \Delta_i$, where*

$$|\Delta_i| \leq \sum_{h=1}^{x} \sum_{y=1}^{p/h} (p/xy + 1).$$

A transformation implies that $|\Delta_i| < \sqrt{p} \log p$.

A consequence of Vinogradov's Theorem is

COROLLARY. $\alpha_i \leq k(\sqrt{p} \log p + 1)$ *for $1 \leq i \leq k - 1$, where p is any odd prime.*

In 1952 Davenport and Erdös [3] improved the results of this corollary by proving.

THEOREM B. *If $k = p_1 p_2 \cdots p_t$ divides $p - 1$, p_i prime and $d(k) = ((k + 1)^{2^t}(2k + 1)(2k^2)^{2^t-2})^{-1}$, then for any $\epsilon > 0$ the inequality $\alpha_i < p^{(1-d(k))/2+\epsilon}$ holds when $p > P_\epsilon$.*

The proof of this theorem depended quite heavily on the Pólya [7]–Vinogradov [8] inequality.

In 1962 Burgess [2] improved the Pólya–Vinogradov inequality in a range critical to the arguments used above by proving

THEOREM C. *If p is a prime, if χ is a non-principal character modulo p,*