ON THE COMPOSITION OF TWO TOURNAMENTS

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1. Introduction. A (round-robin) *tournament* R consists of a nonempty finite set of *nodes* $1, 2, \cdots, n$ such that each pair of distinct nodes i and j is joined by exactly one of the oriented *arcs* **ij** or **ji**. If the arc **ij** is in R we say i *dominates* j and write $i \rightarrow j$; more generally, if every node of a subtournament A dominates every node of a subtournament B we write $A \rightarrow B$.

If the tournaments R and S have r and s nodes, respectively, then the *composition* of R and S is the tournament $R \circ S$ with rs nodes (i, k), where $1 \leq i \leq r$ and $1 \leq k \leq s$, such that $(i, k) \rightarrow (j, l)$ if and only if $i \rightarrow j$ in R or $i = j$ and $k \rightarrow l$ in S. In other words, $R \circ S$ is obtained by replacing each node i of R by a copy S_i of S and letting $S_i \rightarrow S_j$ in $R \circ S$ if and only if $i \rightarrow j$ in R. Notice that for any set $\{S_i : i \in I\}$ of copies of S in $R \circ S$, there exists a subtournament C_I of R such that $\cup_{i \in I} S_i = C_I \circ S$.

The composition of two tournaments is not commutative, in general, and one of our objects here is to characterize those pairs of tournaments R and S for which $R \circ S = S \circ R$. (Two tournaments are equal, or *isomorphic*, if and only if there exists a one-to-one dominance-preserving correspondence between their nodes.) In order to do this it seems necessary to develop various algebraic properties of the composition operation and another operation that is introduced in §2. Sabidussi [7], Hemminger [3], Lovász [4], and others have investigated algebraic properties of other operations on other classes of graphs.

2. Sums of tournaments. If there exist two disjoint subtournaments A and B of the tournament R such that $A \rightarrow B$ and every node of R belongs to A or B, we say that R is the *sum* of A and B and write $R = A + B$. A tournament is *reducible* or *irreducible* according as it can or cannot be expressed as the sum of two smaller tournaments. A tournament R is *strongly connected* if for every ordered pair of nodes p and q there exists a path from p to q, that is, a sequence of arcs in R of the type **pr**, **rs**, $\cdots$, **wq**. It is easy to show (see [6]) that a tournament is irreducible if and only if it is strongly connected.

We shall denote the number of nodes in the tournaments $A, B, \cdots, Z$ by $|A|, |B|, \cdots, |Z|$. Subscripts will serve merely to distinguish one tournament from another except that the symbol T_n $(n = 1, 2, \cdots)$ will always denote the *transitive* tournament with n nodes, that is, the tournament whose n nodes can be labelled in such a way that $i \rightarrow j$ if and only if $i > j$ for $1 \leq i, j \leq n$. Notice that the trivial tournament T_1 is both irreducible and transitive.

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