AN ELEMENTARY NOTE ON CHARACTER SUMS

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Let $\epsilon > 0$, m be a positive integer, χ be a nonprincipal character (mod m), $0 \leq a < b$, and $b^2 - a^2 \leq m$. Set $n = [2^{1/\epsilon}]$ and let $\varphi_2(m)$ denote Jordan's generalization of Euler's function.

By elementary methods, A. Schinzel and M. Bhaskaran, [1], have proved that real χ satisfy:

$$1) \quad \left| \sum_{k=a+1}^{b} \chi(k) \right| < \{\varphi(m) - (12m^2/(\pi^2\varphi_2(m)) - 1)(\varphi(m)/m)^2(b-a)^2\}^{\frac{1}{2}} + O(m^\epsilon).$$

(The factor $m^2/\varphi_2(m)$ does not occur in [1], but the possibility of its introduction, as well as the existence of [1], was pointed out to the author by Professor Schinzel.)

The estimate 1), while not as powerful as the recent one obtained by Burgess [3], is acquired in a simpler manner, and is in some instances stronger than an earlier estimate due to Landau [4].

Without knowing of [1], the author, using methods which were, in part, the same as those of Schinzel and Bhaskaran, obtained an estimate for the density of the cosets of the d-th power residues (mod p) (p an odd prime) in certain intervals. [5].

The main result of this paper is to combine the methods of [1] and [5], and, by so doing, present an estimate such as 1), without the requirement that χ be real. In some special cases, better estimates will be obtained. A certain generalization of the result of [5] will be obtained in the process.

The following two lemmas are proved as in [1].

Lemma 1. *The number q of integers x such that $a < x \leq b$, $(x, m) = 1$, is given by:*

$$q = (b-a)\varphi(m)/m + \delta_1 ,$$

where $1 - 2^{n-1}m^\epsilon/(n!)^\epsilon \leq \delta_1 \leq 2^{n-1}m^\epsilon/(n!)^\epsilon$.

Lemma 2. *For the cardinality C of the set S of all pairs $\langle x, y\rangle$ such that $a < x \leq b$, $a < y \leq b$, $(x, y) = 1 = (xy, m)$, we have:*

$$C = 6\{\varphi(m)(b-a)\}^2/(\pi^2\varphi_2(m)) + \delta_{11} ,$$

where

$$-m^\epsilon(b-a)(b^\epsilon + 2\log b)2^{n-1}/(n!)^\epsilon - 2 \leq \delta_{11} \leq m^\epsilon(b-a)(b^\epsilon + 2\log b)2^{n-1}/(n!)^\epsilon + 1.$$

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