ISOMETRIES OF TRANSLATION-INVARIANT SUBSPACES OF $C(R^n)$

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1. Introduction. The Banach space of bounded continuous complex-valued functions on R^n in the supremum norm is denoted by $C(R^n)$, and the subspace of all $f \in C(R^n)$ which vanish at infinity is denoted by $C_0(R^n)$. Let Λ be a compact subset of R^n, and let $E(\Lambda)$ be the subspace of $C(R^n)$ consisting of the functions with spectrum contained in Λ. In this paper the linear isometries of $E(\Lambda)$ onto itself and of $C_0(R^n) \cap E(\Lambda)$ onto itself are characterized under the assumption that $C_0(R^n) \cap E(\Lambda)$ is nontrivial. Each function belonging to $E(\Lambda)$ can be extended to an entire function of n complex variables. The proof of the characterization of the isometries depends on imbedding $E(\Lambda)$ in the space of continuous functions on a suitably constructed compactification of R^n and on elementary properties of entire functions of several variables. Similar characterizations of isometries of other Banach spaces of entire functions are indicated in §5. Also, the reader is referred to [4] where de Leeuw characterizes the isometries of a Banach space of Lipschitz functions on the real line.

2. Definitions and preliminaries. We shall regard R^n as the subset of C^n consisting of all $z = (z_1, \cdots, z_n) \in C^n$ for which $z_1, \cdots, z_n$ are real. Thus, if f is an entire function of n complex variables, we may speak of its restriction to R^n.

We shall need some terminology and results from harmonic analysis in R^n. Let L^1 be the Banach space of complex-valued Lebesgue integrable functions g on R^n with the norm $\|g\|_1 = \int |g(x)|\, dx$, and let L^∞ be the space of complex-valued bounded Lebesgue measurable functions f on R^n with the norm $\|f\|_\infty = \operatorname{ess\,sup} |f(x)|$. The Banach space L^∞ is the dual of L^1 where the result of the operation of $f \in L^\infty$ as a linear functional on $g \in L^1$ is $\int f(x)g(x)\, dx$. As usual we refer to the topology of L^∞ induced by L^1 as the w^* topology. For any $\xi \in R^n$ write $e(\xi)$ for the exponential function whose value at any $x \in R^n$ is $e(\xi, x) = e^{i\xi\cdot x}$ where

$$\xi\cdot x = \xi_1 x_1 + \cdots + \xi_n x_n .$$

The Fourier transform of any $g \in L^1$ is then

$$\hat{g}(\xi) = \int e(-\xi, x) g(x)\, dx.$$

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