SYMMETRY IN COMPLEX INVOLUTORY BANACH ALGEBRAS II

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1. Introduction. Kaplansky [4] has raised the following question: If a complex Banach algebra has hermitian real-linear involution, is it symmetric? Theorem 1 of [8] asserts that if the involution is continuous and conjugate-linear, then the answer is in the affirmative. Unfortunately there is an error in the proof of Lemma 5, on which the proof of the theorem depends. This error, which will be corrected below, arises as follows. In the course of the argument an appeal is made to the last paragraph of the proof of Proposition I on page 304 of [6], and this proof appears to be invalid. (It is tacitly assumed that if I is a proper left ideal in a symmetric Banach *-algebra R with *-radical R_1 and identity e, then the image of I under the natural homomorphism of R onto R/R_1 is proper. But if I is a maximal left ideal, this will be the case only if I contains R_1, and this last fact is obtained in Proposition II (loc. cit.) as a deduction from Proposition I. This argument is therefore circular. We cannot see how to justify this assumption without using the symmetry of the involution. See [7; 230-9] for valid proofs of these Propositions.)

In the present paper, this error is corrected, and the requirement that the involution be continuous and conjugate-linear is dropped.

2. Notation and definitions. Let A be an algebra over the complex field with real-linear involution $x \to x^*$. The spectrum and spectral radius of an element x of A will be denoted by $sp(x)$ and $\rho(x)$ respectively. The set of all hermitian elements (i.e. all x such that $x^* = x$) will be denoted by H. For elements h and k of H, we write $h > k$ (resp. $h \geq k$) or $k < h$ (resp. $k \leq h$) to indicate that $sp(h - k) \subset [0, \infty)$ (resp. $(0, \infty)$). The involution is *hermitian* if each hermitian element has real spectrum, and *symmetric* if each element of the form x^*x has nonnegative real spectrum. An algebra with hermitian (resp. symmetric) involution is said to be *C-symmetric* (resp. *symmetric*).

3. The main result in this section is

Theorem 1 (Shirali). *Let A be a complex Banach algebra with hermitian conjugate-linear involution. Then A is symmetric.* (Theodore Palmer has extended Theorem 1 to the case in which A is a real Banach algebra with hermitian and skew-hermitian involution.)

The proof will be advanced by a series of lemmas. Note that A is not required to possess an identity element. However, it is always possible to extend * to a conjugate-linear involution on A_1, the algebra obtained by adjoining an

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