EXAMPLES OF INVARIANT SUBSPACE LATTICES

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1. Introduction. If A is a bounded linear operator on a separable infinite-dimensional complex Hilbert space, then the collection of all (closed) subspaces invariant under A is a complete lattice under the partial ordering defined by inclusion. The trivial subspaces, the zero subspace and the whole space, are invariant under every operator A.

Definition. An abstract lattice L is *attainable* if there exists a bounded linear operator A on a separable infinite-dimensional complex Hilbert space such that the lattice of invariant subspaces of A is order-isomorphic to L.

The invariant subspace problem, the problem of whether or not every operator has a non-trivial invariant subspace, is the question of whether or not the lattice 2 is attainable. This problem is still unsolved although many partial results have been obtained. In order to investigate other problems related to the invariant subspace problem, (e.g. the question of whether or not commuting operators have common invariant subspaces), it is helpful to have examples of attainable lattices. The number of known attainable lattices is surprisingly small, and the main purpose of this paper is to add other examples.

It is easy to find the invariant subspaces of some (but not most) normal operators. Beurling initiated the study of lattices of invariant subspaces of non-normal operators in his famous paper [3], where he studied the unilateral shift operator, (see also [9]). We shall not discuss the unilateral shift at all in this paper. However we shall use facts about the invariant subspaces of certain operators related to the shift.

Definition. An operator A on H is a *Donoghue operator* if there is an orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H such that $Ae_0 = 0$, $Ae_j = a_j e_{j-1}$ for $j > 0$, and such that $\{a_j\}$ is a square-summable sequence with non-zero terms that are monotone decreasing in absolute value.

It has been shown [6], [12], [7] that the non-trivial invariant subspaces of a Donoghue operator are the subspaces $M_n = \vee_{j=0}^{n} \{e_j\}$, $n = 0, 1, 2, \cdots$. Thus the lattice of a Donoghue operator is the ordinal number $\omega + 1$.

Another operator whose invariant subspaces are known (see [5], [11], [6], [10], [17]) is the *Volterra operator* V, defined on $\mathfrak{L}^2(0, 1)$ by

$$(Vf)(x) = \int_0^x f(t)\, dt.$$

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