AN EXTENSION OF NEUMANN'S INTEGRALRELATION FOR GENERALIZED LEGENDRE FUNCTIONS

By H. S. V. de Snoo

In this paper we obtain an integralrelation connecting the two linearly independent generalized Legendre functions of Kuipers and Meulenbeld. The result is a generalization of F. Neumann's relation of 1848 for the two kinds of Legendre functions

$$Q_k(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_k(x)}{z - x}\, dx$$

where k is a nonnegative integer, and z is not lying on the segment $(-1, 1)$ of the complex plane.

The main result is in §2; generalizations can be found in §4. E. R. Love's integralrelations of 1965 for associated Legendre functions follow as special cases.

1. The generalized Legendre functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, two specified linearly independent solutions of the differential equation

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\}w = 0,$$

have been introduced by Kuipers and Meulenbeld [3] as functions of z for all points of the z-plane, in which a cross-cut exists along the real x-axis from 1 to $-\infty$, and for complex values of the parameters k, m and n. On the segment $-1 < x < 1$ of the cross-cut these functions are defined in [7]. If $m = n$, they reduce to the associated Legendre functions, defined in [2].

For the sake of brevity we put

$$\alpha = k + \tfrac{1}{2}(m+n), \qquad \beta = k - \tfrac{1}{2}(m-n),$$
$$\gamma = k + \tfrac{1}{2}(m-n), \qquad \delta = k - \tfrac{1}{2}(m+n).$$

Generalized Legendre functions can be written in terms of hypergeometric functions, such as [4, (9)]

$$(1) \qquad Q_k^{m,n}(z) = e^{\pi i m} 2^{\beta} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)} (z+1)^{-k+\frac{1}{2}m-1}(z-1)^{-\frac{1}{2}m} \cdot F\left(\beta+1, \delta+1; 2k+2; \frac{2}{1+z}\right)$$

if z is not lying on the cross-cut.

Received March 7, 1968. The author wishes to express his indebtedness to Professors R. L. van de Wetering and B. Meulenbeld.