UNIQUENESS CLASSES FOR PERIODIC-TYPE FUNCTIONALS

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1. Introduction. Let K denote the set of all entire functions of exponential type; i.e., entire functions for which there exist real numbers A and T such that for all z, $|f(z)| \leq A \exp(T|z|)$. These are the entire functions having a Laplace (Borel) transform analytic and zero at infinity. Given a simply connected domain Ω in the complex plane, let $K[\Omega]$ denote the class of all f in K whose Laplace transforms are analytic on Ω', the complement of Ω. We shall use F to denote the Laplace transform of f.

We are concerned with finding uniqueness classes for a sequence $\{L_n\}$ of linear functionals defined on a set $K[\Omega]$ by

$$L_n(f) = \frac{1}{2\pi i} \int_\Gamma g_n(\zeta) F(\zeta)\, d\zeta \tag{1}$$

where g_n is analytic on Ω and $\Gamma \subset \Omega$ is a simple closed curve such that F is analytic outside and on Γ, for sequences $\{g_n\}$ having a certain periodic nature. A class $K[\Omega]$ is a uniqueness class for $\{L_n\}$ if the zero function is the only function f in $K[\Omega]$ with the property that $L_n(f) = 0$; $n = 0, 1, \cdots$. Much work has been done on this problem for particular sequences $\{g_n\}$ and for certain classes of such sequences; e.g., $g_n(\zeta) = [W(\zeta)]^n$ for some function W[1], [2], [3], [4], [5], [7].

The principal method of attacking the problem has been by use of interpolation series. If a sequence of functions $\{p_n\}$ can be found such that if $\sum_{n=0}^{\infty} L_n(f) p_n(z)$ converges, it converges to f, and if a class $K[\Omega]$ can be found such that for all f in $K[\Omega]$ the series converges on some domain, then obviously $K[\Omega]$ is a uniqueness class for $\{L_n\}$. Stronger uniqueness results can be obtained by replacing convergence by any regular method of summation. The disadvantage of this method is that convergence or summability depend on the shape of the domain Ω or of some domain obtained from it. In an earlier paper [4], the author showed a simple method of obtaining a necessary and sufficient condition for a set $K[\Omega]$ to be a uniqueness class for the case $g_n(\zeta) = [W(\zeta)]^n$ without using interpolation series. The purpose of this paper is to show that this method can also be applied to the more general situation of sequences given by $g_{pn+k}(\zeta) = [W(\zeta)]^{pn} h_k(\zeta)$, $k = 0, 1 \cdots, p-1$; $n = 0, 1, 2, \cdots$.

Classical examples of these sequences of functionals are the Lidstone functionals $L_{2n}(f) = f^{(2n)}(0)$, $L_{2n+1}(f) = f^{(2n)}(1)$, and generalizations of these associated with the names Gontcharoff [6] ($L_{pn+k}(f) = f^{(pn+k)}(a_k)$) and Poritsky [8]

Received March 16, 1968.