NOTES ON PLANE PARTITIONS III

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1. Introduction. Let $b_k(n)$ be the number of k-rowed partitions of n whose non-zero parts decrease strictly along each row, and put $b(n) = \lim_{k\to\infty} b_k(n)$. In [2] it was shown that the generating functions

$$B_k(x) = \sum_{n=0}^{\infty} b_k(n)x^n \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} b(n)x^n \text{ are given by}$$

$$B_k(x) = P^{[k/2]}Q^{2\{k/2\}} \prod_{\nu=1}^{k-2} (1 - x^\nu)^{[(k-\nu)/2]}, \tag{1}$$

$$B(x) = \prod_{\nu=1}^{\infty} (1 - x^\nu)^{-[(\nu+1)/2]}, \tag{2}$$

where

$$P = \prod_{\nu=1}^{\infty} (1 - x^\nu)^{-1}, \; Q = \prod_{\nu=1}^{\infty} (1 - x^{2\nu-1})^{-1},$$

and where $[\theta]$, $\{\theta\}$ denote the integral and fractional parts of θ. In this note we will make use of these expressions to obtain asymptotic formulas for $b_k(n)$ and $b(n)$. In the case of $b_k(n)$, we derive a convergent series expansion of the type of Hardy–Ramanujan–Rademacher [4], [8] for the ordinary partition function. In the case of $b(n)$ this is no longer possible since $B(x)$ is not the product of a algebraic function and a modular form. Nevertheless, by using the method of E. M. Wright [10], who solved the corresponding problem for $a(n)$, the total number of plane partitions of n, we will obtain an asymptotic expansion. The dominant terms of these series give the asymptotic formulae

$$b_k(n) \sim 2^{-1-\frac{1}{2}[\frac{1}{2}k]-\{\frac{1}{2}k\}}\pi^{k^2/4-\frac{1}{2}k+\frac{1}{2}\{\frac{1}{2}k\}}(k/12n)^{(k^2-k+2)/8}\, e^{\pi(kn/3)^{\frac{1}{2}}}, \prod_{j=1}^{[\frac{1}{2}(k-1)]} (k - 2j)!$$

and

$$b(n) \sim 2^{-\frac{3}{4}}(3\pi\zeta(3))^{-\frac{1}{2}}N^{-49/24}e^{\frac{3}{2}\zeta(3)N^2+\pi^2N/24-\pi^4/3456\zeta(3))+c}$$

where

$$c = \int_0^\infty \frac{y \log y\, dy}{e^{2\pi y} - 1},$$

$\zeta(s)$ is the Riemann ζ-function, and $N = (n/\zeta(3))^{\frac{1}{3}}$.

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