THE ABSTRACT F. AND M. RIESZ THEOREM

By Heinz König and G. L. Seever

1. Introduction and measure theoretic preliminaries. The chain of abstract F. and M. Riesz theorems which issued from the work of Helson and Lowdenslager [9] (via Bochner's observation [3] of the generality of their arguments) was completed by the following

Theorem. (Ahern [1]) *Let X be a compact Hausdorff space, A a sup norm algebra on X, φ a non-zero multiplicative linear functional on A, and let λ be a representing measure for φ. Then the following conditions are equivalent.*

(a) *If $\mu \in A^{\perp}$ (= set of all complex Baire measures μ on X such that $\int f\, d\mu = 0$ for all $f \in A$), then μ_λ, the λ-continuous part of μ, also belongs to $A^{\perp}$.*

(b) *Every representing measure for φ is λ-continuous.*

In 1967 Glicksberg [6] gave a "universal" F. and M. Riesz theorem. Let $M(A, \varphi)$ be the set of representing measures for φ (we retain the notation of Ahern's theorem). Glicksberg defines a Baire set $E \subset X$ to be φ-null iff $\mu(E) = 0$ for all $\mu \in M(A, \varphi)$ and then notes that $M(X)$ (= space of complex Baire measures on X) is the direct sum of the space of φ-continuous measures, i.e., those $\mu \in M(X)$ such that $\mu(E) = 0$ whenever E is φ-null, and the space of φ-singular measures, i.e., those $\mu \in M(X)$ such that μ lives on a φ-null set (cf. [8; 42]). His abstract F. and M. Riesz theorem states that if $\mu \in A^{\perp}$, then the φ-continuous and φ-singular parts of μ also belong to $A^{\perp}$. If (b) of Ahern's theorem holds, then φ-continuity (resp., φ-singularity) is just λ-continuity (resp., λ-singularity) so that Ahern's theorem is a corollary of Glicksberg's.

In this paper we consider the following situation: X is a set, Σ is a σ-algebra of subsets of X, A is an algebra of bounded, Σ-measurable, complex-valued functions on X and $1 \in A$, $A^{\perp}$ is the set of $\mu \in ca(x, \Sigma)$ (= space of countably additive complex-valued functions on Σ; briefly, measures) such that $\int f\, d\mu = 0$ for all $f \in A$. For $\varphi \in S(A)$, the set of non-zero multiplicative linear functionals on A, $M(A, \varphi)$ denotes the set of all probability measures μ on Σ such that $\varphi(f) = \int f\, d\mu$ for all $f \in A$. For $\varphi \in S(A)$ we exhibit a direct sum decomposition of $ca(X, \Sigma)$ such that

(i) for $\mu \in ca(X, \Sigma)$, one composant is λ-continuous for some $\lambda \in M(A, \varphi)$, and the other composant is λ-singular for all $\lambda \in M(A, \varphi)$,

(ii) (Abstract F. and M. Riesz theorem) If $\mu \in A^{\perp}$, then the composants of μ also belong to $A^{\perp}$.

Received February 1, 1968. Partly supported by National Science Foundation Grant GP-6111.