A GENERALIZATION OF FREE ACTION

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0. Introduction. In [5] von Neumann defined the notion of free action and applied it to the construction of certain factors (c.f. also Dixmier, [3; 127–137]). Given an abelian von Neumann algebra $\mathcal{R}$ and a *-automorphism φ of $\mathcal{R}$, von Neumann defined φ to be freely acting on $\mathcal{R}$ if:

(0.1) Given a projection P in $\mathcal{R}$, there exists a nonzero projection Q in $\mathcal{R}$ with $Q \leq P$ and $\varphi(Q) \perp Q$.

This paper is chiefly devoted to the study of an extension of this notion to general von Neumann algebras.

In §1 a definition of free action for general $\mathcal{R}$ is given. There this notion is studied in some detail. Along the way a few results on outer automorphisms of von Neumann algebras are derived. The chief result of §1 is a structure theorem. Given an arbitrary von Neumann algebra $\mathcal{R}$ and an arbitrary *-automorphism φ of $\mathcal{R}$, then $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $\varphi = \varphi_1 \oplus \varphi_2$, where $\varphi_i(\mathcal{R}_i) \subseteq \mathcal{R}_i$ $(i = 1, 2)$, φ_1 is inner on $\mathcal{R}_1$, and φ_2 is freely acting on $\mathcal{R}_2$. This decomposition is unique.

§2 is devoted to the study of certain automorphisms of the left ring of a discrete group. The results in this section, even though easy, apparently have not been noticed before.

Two applications of the results of §§1 and 2 are given in §3.

This notation of this paper is essentially that of Dixmier [3]. The exceptions are the following. If $\mathcal{R}$ is a von Neumann algebra, $P(\mathcal{R})$ denotes the lattice of all projections of $\mathcal{R}$, $U(\mathcal{R})$ denotes the unitary group of $\mathcal{R}$, and $SA(\mathcal{R})$ denotes the real linear space of self-adjoint elements of $\mathcal{R}$. If $T \in \mathcal{R}$, $S(T)$ denotes the support of T [3; 333] and $C(T)$ denotes the central support of T (Dixmier, [3; 7].)

1. The notion of free action. First a result on outer automorphisms of operator algebras.

Theorem 1.1. *Let $\mathcal{R}$ be a von Neumann algebra and φ a *-automorphism of $\mathcal{R}$. Then φ is outer if and only if φ satisfies the following condition:*

(1.1) *Given $A \in \mathcal{R}$ such that $AB = \varphi(B)A$ for all $B \in \mathcal{R}$, then $C(A) < I$.*

Proof. Suppose (1.1) holds and φ is inner. Then there exists $U \in U(\mathcal{R})$ such that $\varphi(B) = UBU^*$ for all $B \in \mathcal{R}$. Hence $UB = \varphi(B)U$ for all $B \in \mathcal{R}$, and therefore $C(U) < I$. But U unitary implies $C(U) = I$. Contradiction. Hence φ is outer.

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