# FREE ACTIONS OF $Z_4$ ON $S^3$

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**Introduction.** In [1], G. R. Livesay proves that every fixed-point-free homeomorphism on $S^3$ with period 2 is equivalent to the antipodal map. The object of this paper is to prove the

THEOREM. *Every free action of $Z_4$ on $S^3$ is equivalent to the orthogonal action.*

$Z_4$ is the cyclic group of order 4 and to say that it acts freely on $S^3$ means that each $g \in Z_4$ is a fixed-point-free homeomorphism of $S^3$. This theorem characterizes those 3-manifolds which have $S^3$ as universal covering space and $Z_4$ as fundamental group.

**Equivariant general position.**

LEMMA. *If the finite group $G$ acts freely and simplicially on the closed, combinatorial manifold $M$ and $P$ is a subpolyhedron of $M$ invariant under a subgroup $H$ of $G$, then there is an arbitrarily small isotopy of $M$ which takes $P$ onto $Q$ such that for each $g \in G/H$, $Q$ is in general position with respect to $g(Q)$. Moreover, $Q$ is invariant under $H$.*

This is probably well known. The proof consists of using general position isotopies in small neighborhoods, and copying that isotopy in the images of those neighborhoods under $G$. An equivariant subdivision may be necessary at the beginning.

**Proof of the theorem.** Since $Z_4$ acts freely on $S^3$, $S^3/Z_4$ is a closed manifold and $S^3$ is its universal covering space. $S^3/Z_4$ may be triangulated and this triangulation lifted to $S^3$. The action of $Z_4$ on $S^3$ is the deck transformation of the covering space, and hence is simplicial. Let $T \in Z_4$ be a generator. Then $T^2$ is an involution of $S^3$ without fixed points and is, by [1], equivalent to the antipodal map. It will be assumed, therefore, that $T_2$ is the antipodal map. Let $S$ be a locally flat, polyhedral 2-sphere in $S^3$ invariant under $T^2$. By the lemma, it is assumed that $S$ is in general position with respect to $TS$. Then $S \cap TS$ is a collection $C = C_1 \cup \cdots \cup C_n$ of disjoint simple closed curves. $n \neq 0$ by the following argument: Let $E^+$ and $E^-$ be the closed complimentary domains of $S$ in $S^3$. $n = 0$ means $TS \subset \text{int } E^+$ or $TS \subset \text{int } E^-$. If $TS \subset \text{int } E^+$ then $TE^+ \subset \text{int } E^+$ or $TE^- \subset \text{int } E^+$. $TE^+ \subset \text{int } E^+$ implies $T^2E^+ = E^- \subset TE^+ \subset \text{int } E^+$ which is a contradiction. $TE^+ \subset \text{int } E^-$ implies $T^2E^- = E^+ \subset T(\text{int } E^+)$, so $TE^- \subset T(\text{int } E^+)$ so $E^- \subset \text{int } E^+$, which is also a contradiction. Similarly $TS$ cannot be contained in int $E^-$, therefore $S \cap TS \neq \varnothing$. $C$ divides

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