UNIFORM APPROXIMATION OF RATIONAL FUNCTIONS BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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In this paper we give a necessary and sufficient condition in order that a rational function be uniformly approximable on a class of compact subsets of the complex plane by polynomials whose coefficients are, in a certain sense, integers.

We assume throughout the paper that A is any discrete subring of the complex numbers $\mathbf{C}$ with rank 2 and unique factorization. For example, A could be the Gaussian integers $\mathbf{Z} + i\mathbf{Z}$, where $\mathbf{Z}$ denotes the rational integers. We say that a function is A-approximable on a set X if it is uniformly approximable on X by elements of $A[z]$.

We call a compact subset of the complex plane Mergelyan if it has the property that any continuous complex valued function on X which is holomorphic on X° (the interior of X) can be uniformly approximated by polynomials. This is equivalent to requiring that X have connected complement [3] or that X be polynomially convex. The requirement that X be Mergelyan is no real restriction since a function which is A-approximable on a compact subset X of $\mathbf{C}$ has an extension to the polynomial convex hull of X which is also A-approximable [2, §2]. Throughout the paper we suppose X to be any Mergelyan subset of the open unit disk D° such that $0 \in X^\circ$. It is easy to see, however, that if we translate X by an element of A the theorem remains valid. The main result of the paper is the following.

Theorem. *A rational function f is A-approximable on X if and only if it can be represented in the form $f = p/g$ where p and g are in $A[z]$, $g(0)$ is a unit of A, and the roots of g lie outside of X.*

Proof. First suppose that f is represented as in the theorem. Then f is continuous on X and holomorphic on X° so by Ferguson [2, 4.8] it suffices to prove that the coefficients of the power series expansion

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{1}$$

lie in A. Let

$$p(z) = \sum_{k=0}^{n} a_k z^k \tag{2}$$

and

$$g(z) = \sum_{k=0}^{n} b_k z^k. \tag{3}$$

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