ON ZETA FUNCTIONS OF NUMBER FIELDS

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Let K be a number field of degree n over $\mathbf{Q}$ and let $\mathfrak{J}$ be the ring of integers of K. Then the zeta function of K is defined by $\mathrm{Z}(s) = \sum N\mathfrak{a}^{-s}$ where the sum ranges over the integral ideals of $\mathfrak{J}$ and where the norm $N\mathfrak{a}$ is the group-index $[\mathfrak{J} : \mathfrak{a}]$. This series converges absolutely for Re $(s) > 1$ and is equal to the Euler product $\prod (1 - N\mathfrak{P}^{-s})^{-1}$ where $\mathfrak{P}$ ranges over the maximal ideals of $\mathfrak{J}$. Furthermore, $\mathrm{Z}(s)$ satisfies a functional equation and can be extended to a meromorphic function in the whole plane [3]. In this paper we give some preliminary indications of what happens for non-maximal orders of K. The main reason for doing so is the occurrence of such orders in the theory of complex multiplication.

By an *order* in K is meant a subring $\mathfrak{O}$ of K which contains 1 and is a finite $\mathbf{Z}$-module containing a $\mathbf{Q}$-basis of K. Each such order is contained in the unique maximal order $\mathfrak{J}$. It is known that the series $\sum N\mathfrak{a}^{-s}$, where $\mathfrak{a}$ ranges over the integral $\mathfrak{O}$-ideals and $N\mathfrak{a} = [\mathfrak{O} : \mathfrak{a}]$, converges absolutely for Re $(s) > 1$, [2]. Also, as we show below, the Euler product for $\mathfrak{O}$ converges in the same domain. However, in the non-maximal case the two functions do not coincide, and so there is a question as to which one should be taken as the definition of the zeta function. There are strong reasons for choosing the Euler product. In particular, this would coincide with the definition for the zeta function of the affine scheme Spec $\mathfrak{O}$ that is customary in algebraic geometry [4]. It follows from known results for preschemes of finite type over $\mathbf{Z}$ that the Euler product in the present case will converge absolutely for Re $(s) > 1$. For the sake of completeness, we give a more elementary proof which is only a slight variation on the classical method used for $\mathfrak{J}$. We define the *zeta function* $\zeta(s)$ of $\mathfrak{O}$ to be the product $\prod (1 - N\mathfrak{p}^{-s})^{-1}$ where $\mathfrak{p}$ ranges over the maximal ideals of $\mathfrak{O}$.

Proposition. *The zeta function $\zeta(s)$ of $\mathfrak{O}$ converges absolutely for* Re $(s) > 1$.

Proof. Recall that an infinite product $\prod_{k=1}^{\infty} (1 + w_k)$ converges absolutely if and only if $\sum_{k=1}^{\infty} |w_k|$ converges; furthermore the product will then converge to a non-zero value provided $1 + w_k \neq 0$ for all k, and then $\prod_{k=1}^{\infty} (1 + w_k)^{-1}$ also will converge absolutely. Consequently it suffices to establish the absolute convergence of the product $\prod (1 - N\mathfrak{p}^{-s})$ for Re $(s) > 1$. To do this, we show that $\sum_{\mathfrak{p}} N\mathfrak{p}^{-s}$ converges for s real and $s > 1$. If $\mathfrak{p}$ is a maximal ideal of $\mathfrak{O}$ then $\mathfrak{p} \cap \mathbf{Z} = p\mathbf{Z}$ for some rational prime p. Then $N\mathfrak{p}$ is divisible by p. To see this, set $m = N\mathfrak{p}$ and let $\alpha_1, \cdots, \alpha_m$ be a complete residue system for $\mathfrak{O}$ mod $\mathfrak{p}$. Then so is $\alpha_1 + 1, \cdots, \alpha_m + 1$. Therefore $\sum_{i=0}^{m} (\alpha_i + 1) - \sum_{i=0}^{m} \alpha_i = m$ is an

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