SOME REMARKS ON CONVOLUTION OPERATORS AND $L(p, q)$ SPACES

By Leonard Y. H. Yap

1. Introduction. The main purpose of this paper is to give an elementary and straightforward proof of the crucial Lemma (1.6) in O'Neil [2], thus leading to a very simple proof of the often quoted "convolution theorem" in that paper [see Theorem (3.4) below]. In fact, the proof of Lemma (1.5) (on which the lemma cited above depends) in [2] is incomplete, and it seems impossible to justify some of the inequalities involved because the convolution operator (as defined in [2, (2.1)]) lacks an important continuity property. For this reason, we propose an alternative definition of a convolution operator (called *"positive convolution operator"* in Definition (2.3) below) and then show that it can be extended in such a manner that the desired continuity property is available. The main result in O'Neil [2] is valid for ordinary convolution of functions defined on *unimodular* locally compact groups. Our formulation is also guided by this special case, even though our final (rather abstract) result is more general and it yields a corollary which is valid for *all* locally compact groups. Furthermore, for the case of unimodular groups, our result improves that of O'Neil [2] considerably. The precise statements and proofs of these facts are given in §§2 and 3. In §4, we use the ideas developed earlier to prove a new convolution theorem for $L(p, q)$ spaces on locally compact groups (see Theorem (4.1)) which generalizes a well known theorem for L_p spaces. This generalization is similar to O'Neil's generalization of W. H. Young's theorem.

If the reader is interested only in our proof of O'Neil's convolution theorem for unimodular locally compact groups, it suffices to read items (2.1), (2.2), and (2.7) through (3.4), and interpret the term "positive convolution operator" to mean ordinary convolution of functions on locally compact groups.

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All the functions considered in this paper are assumed to be almost everywhere complex valued, unless the contrary is specified or is clearly dictated by the context.

2. Rearrangement of functions and positive convolution operators.

(2.1) Definitions. Let f be a measurable function defined on a measure space (X, μ). For $y \geq 0$, we define

$$m(f, y) = \mu\{x \in X : |f(x)| > y\}.$$

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