NONEXPANSIVE MAPPINGS AND THE WEAK CLOSURE OF SEQUENCES OF ITERATES

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1. Introduction. Let X be a Banach space and K a nonempty closed convex subset of X. A mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Interest in nonlinear nonexpansive mappings has arisen quite recently, (cf. [1], [2], [4] – [9]). In [5] De Marr proves that if K is compact, then every commutative family of nonexpansive mappings of K into itself has a common fixed-point. In generalizing this theorem, Belluce and the writer [1] employed the assumption that for each $x \in K$, the closure of $\{T^n x\}$ contains a point of some given compact set M. (This assumption has also been used by Göhde [7; 54].) In this paper we make a weaker assumption. We are primarily interested here in nonexpansive mappings T which have the property that for each $x \in K$ the *weak closure* of the sequence $\{T^n x\}$ of iterates contains a point of some given set M. Assumptions on M will vary.

In §2 of this paper we note an improvement of a theorem of [1] and prove some related theorems. An example is given in §3 which shows that certain hypotheses of one of our theorems cannot be removed, and in the next section, the writer's theorem of [8] is extended to a wider class of spaces for mappings which are strictly contractive.

Throughout the paper we use the following notation: X always denotes a Banach space. For a subset A of X, coA and $\overline{co}A$ denote, respectively, the convex hull and the closed convex hull of A; wcA denotes the weak closure of A. For $A, B \subseteq X$, we let

$$\delta(A) = \sup \{\|x - y\| : x, y \in A\};$$

$$d(A, B) = \inf \{\|x - y\| : x \in A, y \in B\}.$$

2. Weak closure of iterates. The following concept was introduced by Brodskii and Milman [3].

Definition 2.1. A bounded convex subset K of X is said to have *normal structure* if for each convex subset H of K which contains more than one point there is a point $x \in H$ which is not a diametral point of H (i.e., $\sup \{\|x - y\| : y \in H\} < \delta(H)$).

A bounded convex subset K of X has normal structure if X is uniformly convex, or if K is compact. (This latter fact is essentially Lemma 1 of [5].)

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