REMARKS ON UNIVERSAL SENTENCES OF $\mathbf{L}_{\omega_1,\omega}$

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The study of the relationship between the language $\mathbf{L}_{\omega_1,\omega}$ and the class $\mathcal{K}_{\omega_1}$ of all countable structures was begun by Dana Scott in [3]. One of the principal results of [3] was the Countable Isomorphism Theorem.

Theorem (Scott). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures. If $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathbf{L}_{\omega_1,\omega}$, then they are isomorphic. In fact, there is a single sentence $\varphi_{\mathfrak{A}}$ of $\mathbf{L}_{\omega_1,\omega}$ such that for any countable structure $\mathfrak{B}$, $\mathfrak{B}$ is a model of $\varphi_{\mathfrak{A}}$ if and only if $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$.*

The purpose of this note is to point out uses of an interpolation theorem of Malitz [2] in the study of the relationship between countable structures and universal sentences of $\mathbf{L}_{\omega_1,\omega}$. For example, our first result, an analogue of Scott's Theorem, gives a necessary and sufficient syntactic condition that a countable structure $\mathfrak{A}$ be embeddable in another countable structure $\mathfrak{B}$. For the most part our notation follows that of [2]. We allow formulas of $\mathbf{L}_{\omega_1,\omega}$ to contain constant and relation symbols, the equality symbol $\approx$, but for simplicity, no function symbols.

Interpolation Lemma (Malitz). *Let φ_0 and φ_1 be sentences of $\mathbf{L}_{\omega_1,\omega}$ such that $\varphi_0 \rightarrow \varphi_1$ is valid, and φ_1 is universal. Then there is a universal sentence θ, containing constant and relation symbols common to φ_0 and φ_1, such that $(\varphi_0 \rightarrow \theta) \wedge (\theta \rightarrow \varphi_1)$ is valid.*

This lemma, as stated in [2], applies only to sentences which do not contain the equality symbol. The extension to the above result is sketched, however.

By a countable structure $\mathfrak{A}$, we mean a countable set A with a countable number of constants and finitary relations. By extending the Interpolation Lemma to the language containing function symbols, as outlined in [2], we could extend all of our results to structures containing functions. $\mathcal{K}_{\omega_1}$ is the class of all countable structures. We refer to Theorem 1 as the Countable Embedding Theorem.

Theorem 1. *For any two countable structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}$ can be embedded in $\mathfrak{B}$ if and only if every universal sentence of $\mathbf{L}_{\omega_1,\omega}$ which holds in $\mathfrak{B}$ also holds in $\mathfrak{A}$.*

Proof. If $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$, then every universal sentence true in $\mathfrak{A}_1$ is true in $\mathfrak{A}_0$. So let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ in $\mathcal{K}_{\omega_1}$ be such that every universal sentence true in $\mathfrak{A}_1$ is true

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