A NECESSARY AND SUFFICIENT CONDITION FOR UNIVALENCE OF A MEROMORPHIC FUNCTION

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1. Introduction. We introduce in this paper a necessary and sufficient condition which is a generalization of the well-known [1] necessary condition for univalence of a function in the unit circle $|z| < 1$, namely:

$$|\{f, z\}| \leq \frac{6}{(1 - |z|^2)^2}, \quad \text{where} \quad \{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of $f(z)$.

2. Definition of "the invariant sequence". Let D be a domain the z-plane, $f(z)$ a meromorphic function in D, and $w \varepsilon D$ a point which is not a pole such that $f'(w) \neq 0$. We define the "First generating function"

$$F(z; w) = \frac{f(z + w) - f(w)}{f'(w)},$$

where z varies in a small circle around w. We also define the "Second generating function"

$$F(z; w)^{-1} = G(z; w) = \frac{f'(w)}{f(z + w) - f(w)}.$$

The expansion of $f(z + w)$ around w is

$$f(z + w) = f(w) + zf'(w) + \frac{z^2}{2!} f''(w) + \cdots + \frac{z^n}{n!} f^{(n)}(w) + \cdots$$

and so

$$F(z; w) = z + z^2 \frac{f''(w)}{2!\, f'(w)} + z^3 \frac{f'''(w)}{3!\, f'(w)} + \cdots + z^n \frac{f^{(n)}(w)}{n!\, f'(w)} + \cdots \tag{2.1}$$

We denote

$$a_n(f; w) = \frac{f^{(n)}(w)}{n!\, f'(w)}, \qquad n = 1, 2, \cdots . \tag{2.2}$$

So we have

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