ENUMERATION OF PERMUTATIONS OF (1, $\cdots$, n) BY NUMBER OF MAXIMA

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In this paper we are interested in counting the number of permutations of $(1, \cdots, n)$ of which k entries a_i satisfy $a_i > a_{i-1}$, $a_i > a_{i+1}$. We consider the circular problem, i.e., $a_0 = a_n$, $a_{n+1} = a_1$. For example the permutation (6, 3, 2, 5, 1, 7, 4) has three maxima, namely 6, 5 and 7.

We first obtain a recurrence relation for $M(n, k)$, the number of permutations of $(1, \cdots, n)$ with k maxima. This recurrence relation leads to a partial differential equation whose solution provides a generating function for $M(n, k)$. This solution is then expanded into a power series from which an explicit formula for $M(n, k)$ is obtained.

Throughout the paper we follow the usual convention that $\binom{a}{b} = 0$ if $b < 0$ or $0 \leq a < b$. Also n will always be positive.

A sequence $(a_1, \cdots, a_n)$ will be called an *n-permutation* if and only if it is a permutation of $(1, \cdots, n)$.

A sequence $(a_1, \cdots, a_{n+1})$ will be called an *extension* of an n-permutation if and only if $(a_1, \cdots, a_n)$ is an n-permutation and $a_{n+1} = n + 1$. For future use we note each $(n + 1)$-permutation may be obtained in exactly one way as a cyclic permutation of an extension of an n-permutation.

A member a_i, $i = 1, \cdots, n$ of an n-permutation is a *maximum* if and only if $a_i > \max(a_{i-1}, a_{i+1})$ where $a_0 = a_n$ and $a_{n+1} = a_1$. $M(n, k)$, $n \geq 1$, will denote the number of n-permutations having exactly k maxima. For convenience we define $M(1, 0) = 1$ and $M(n, 0) = 0$ for $n \geq 2$.

We note the following properties of $M(n, k)$ all of which are obvious:

i) $$\sum_{k=0}^{n} M(n, k) = n!$$

ii) The number of maxima of an n-permutation is invariant under cyclic permutations.

iii) $M(n, k) \neq 0$ only if $n/2 \geq k \geq 1$ or $n = 1$ and $k = 0$.

LEMMA. $M(n + 1, k) = (n + 1)/n[2kM(n, k) + (n - 2k + 2)M(n, k - 1)]$ *for* $n \geq 1, k \geq 1$.

Proof. From property ii) it follows that for fixed i, $1 \leq i \leq n$, the number of n-permutations $(a_1, \cdots, a_n)$ with k maxima one of which is a_i is $(k/n)M(n, k)$. Now each $(n + 1)$-permutation with k maxima is a cyclic permutation of an

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