ON FAKE SOUSLIN TREES

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A Souslin space is a linearly ordered topological space which, when given the open interval topology, is usually conceived to be connected and nonseparable although every collection of disjoint open sets is countable. The first Countability Axiom holds true in a Souslin space (but the second Countability Axiom does not), and every closed interval is compact. Recently models of set theory have been described in which Souslin spaces exist. In certain of these models not only is the axiom of choice true but so is the continuum hypothesis. The discovery of these models gives new interest to the existence of Souslin trees and fake Souslin trees. (These descriptive terms are due to M. E. Rudin [5]. Kurepa referred to them as ramifications [1].)

To understand how such structures arise, consider the following procedure: Suppose that X is a Souslin space which contains no separable interval. Let $H_1, H_2, \cdots, H_z, \cdots$ $(z < \omega_1)$ denote a well-ordered (uncountable) sequence such that H_1 is a maximal collection of disjoint open intervals, H_2 is a maximal collection of disjoint open intervals which refines H_1, and for each countable ordinal z, H_z is a maximal collection of disjoint open intervals such that for each ordinal $x < z$, H_z refines H_x. Let H_z^* denote the union of the elements of H_z. Since $X - H_z^*$ is separable for each countable z and the union of countably many separable sets is separable, it is clear that the sequence $H_1, H_2, \cdots$ may be defined so as to be uncountable. In particular, if one insists that H_{z+1} *properly* refine H_z, then the sequence must necessarily terminate after the first $\aleph_1$ steps. In this case $\bigcap H_z^*$ $(z < \omega_1)$ is empty. For suppose, on the contrary, that there exists a point p such that for each $z(z < \omega_1)$, some element h_z of H_z contains p. Since h_{z+1} is a proper subset of h_z, let u_z denote an open interval lying in $h_z - h_{z+1}$. The collection of all such open intervals u_z would be uncountable and no two of them would intersect. This is impossible in a Souslin space. Hence, $\bigcap H_z^* = \phi$.

In fact, the same sort of argument shows that every chain (monotone collection) of elements of $\bigcup H_z$ is countable.

So if there exists a Souslin space X, then there exists a sequence $H_1, H_2, \cdots, H_z, \cdots$ $(z < \omega_1)$ such that (1) for each $z < \omega_1$, H_z is a countable collection of disjoint sets, (2) for $x < z < \omega_1$, each element of H_x contains an element of H_z and each element of H_z is a proper subset of some element of H_z and (3) $\bigcap H_z^* = \phi$ and every monotone subcollection of $\bigcup H_z$ is countable. If one lets $H_0 = \{X\}$, then the sequence $H_0, H_1, H_2, \cdots, H_z, \cdots$ is called a "tall" tree. Now when a "tall" tree is constructed in this manner from a Souslin

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