ON THE ASYMPTOTIC VALUES OF A HOLOMORPHIC FUNCTION WITH NONVANISHING DERIVATIVE

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Let $w = f(z)$ be a holomorphic function defined in the open unit disc D, and let W denote the extended w-plane. An asymptotic path of f for the value $a \in W$ is a simple continuous curve α: $z(t)$, $0 \leq t < 1$, lying in D such that $|z(t)| \to 1$ and $f(z(t)) \to a$ as $t \to 1$. If in addition $z(t) \to e^{i\theta}$ as $t \to 1$, we say that α ends at $e^{i\theta}$ and that f has the asymptotic value a at $e^{i\theta}$. MacLane [2] has considered the class $\mathfrak{A}$ which he defined as follows: $f \in \mathfrak{A}$ if and only if f is a nonconstant holomorphic function defined in D and $\{e^{i\theta}: f$ has an asymptotic value at $e^{i\theta}\}$ is dense on the unit circumference C. We write $f \in \mathfrak{A}_p$ if and only if $f \in \mathfrak{A}$ and each asymptotic path of f ends at a point. According to Bagemihl and Seidel [1], $\mathfrak{A}_p$ contains the nonconstant normal holomorphic functions. We extend a theorem of MacLane [3, Theorem 9] as follows:

THEOREM. *Suppose $f \in \mathfrak{A}_p$ and $f'(z) \neq 0$. Then for any arc $\gamma \subset C$, there exist distinct points $\zeta_j \in \gamma (j = 1, 2, 3)$ and distinct points $a_j \in W (j = 1, 2, 3)$ such that f has the asymptotic value a_j at $\zeta_j (j = 1, 2, 3)$.*

Remarks. MacLane obtained this same conclusion under the assumptions $f' \in \mathfrak{A}$ and $f'(z) \neq 0$. The modular function shows that the number three in the present theorem is best possible.

Proof. Suppose contrary to the assertion that there exists an open arc γ of C for which no such ζ_j and a_j exist. Then there exists an open arc $\gamma' \subset \gamma$ such that at points of γ', f has at most two asymptotic values. By a theorem of MacLane [2; 28], f has the asymptotic value ∞ at each point of a set that is dense on γ'. Thus there exists a finite complex number a such that f has no finite asymptotic value different from a at any point of γ', and by considering the function $f(z) - a$, we see that we can suppose without loss of generality that $a = 0$. Let α_1 and α_2 be asymptotic paths of f for the value ∞ that end at distinct points ζ_1 and ζ_2 respectively of γ'. Let $\Delta_j(\lambda)$ $(\lambda > 0; j = 1, 2)$ be the component of $\{z: |f(z)| > \lambda\}$ that contains all points of α_j that are sufficiently near ζ_j. Since each asymptotic path of f for the value ∞ ends at a point, there exists $M > 0$ such that if we let $\Delta_j = \Delta_j(M)$, then $\bar{\Delta}_j \cap C \subset \gamma' (j = 1, 2$; the bar denotes closure in the plane). Since Δ_j contains no asymptotic path of f for a finite value, and since the Riemann surface $\mathcal{S}$ over W onto which f maps D has no (algebraic) branch point, we see that f maps Δ_j onto a copy of the universal covering surface of $\{M < |w| < +\infty\}$. Thus since the boundary of Δ_j relative to D contains no asymptotic path of f, we see that it is a single level curve Λ_j which, according to a theorem of Mac-

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