HAAR POLYNOMIALS ON CARTESIAN PRODUCT SPACES

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Introduction. The purpose of this paper is to generalize some of the results contained in *Some Theorems on Čebyšev Approximation*, by D. J. Newman and H. S. Shapiro [3]. The notation is that of [3]. The contents form part of the first author's doctoral dissertation at Yeshiva University, under the supervision of Professor Newman. The dissertation title is *Approximation to Separated Functions on Cartesian Product Spaces.*

1. In [3], Newman and Shapiro are concerned primarily with uniqueness questions arising from Čebyšev approximation on Cartesian product spaces by ordinary polynomials in $x_1, \cdots, x_k$ to functions of form $\sum_{i=1}^{k} F_i(x_i)$.

DEFINITION. A family $\{\varphi^u(x)\}_{u=0,1,\dots}$ of continuous real-valued functions on some compact set X is a *Haar sequence* or satisfies the *Haar condition* if: for any $J \geq 0$, any linear combination $\sum_{u=0}^{J} c_u\varphi^u(x)$ with c_u real and not all zero, has at most J zeros in X. (A Haar sequence is defined in [2; 67 et seq], in which such a family is called a *Tchebycheff system with respect to X.*) Equivalently: $\sum_{u=0}^{J} c_u\varphi^u(x) = 0$ for $x = \xi^1, \xi^2, \cdots, \xi^{J+1}$ distinct points of X *implies*

$$c_u = 0 \quad \text{all} \quad u = 0, \cdots, J.$$

Approximation by linear combinations of such $\varphi^u(x)$ are of special interest because it is well known [5; 87ff] that the Haar condition is necessary for the uniqueness of the best approximation even for functions of one variable.

DEFINITION. If $\{\varphi^u(x)\}_{u=0,\dots}$ is a Haar sequence, a *Haar polynomial* (abbrev. H.p.) is any expression of the form $\sum_{u=0}^{J} c_u\varphi^u(x)$. The *degree* of $\sum_{u=0}^{J} c_u\varphi^u(x)$ is the largest u for which $c_u \neq 0$.

Thus, a H.p. of degree d has at most d distinct zeros; and if two H.p. of degree $\leq d$ agree at $d + 1$ points, they are identical.

Assume $\{\varphi^u(x)\}_{u=0,\dots}$ is a Haar sequence on X. The proofs of the following Lemmas are immediate, by standard theorems on existence and uniqueness of solutions to systems of linear equations. [1; ch. II].

LEMMA 1.1. *If $\xi^1, \cdots, \xi^{J+1}$ are distinct values of x, then*

$$\begin{vmatrix} \varphi^0(\xi^1) & \varphi^1(\xi^1) & \cdots & \varphi^J(\xi^1) \\ \vdots & \vdots & & \vdots \\ \varphi^0(\xi^{J+1}) & \varphi^1(\xi^{J+1}), & \cdots & \varphi^J(\xi^{J+1}) \end{vmatrix} \neq 0.$$

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