DISCONTINUOUS FUNCTIONS OF BOUNDED VARIATION AND A NEW CHANGE OF VARIABLE THEOREM FOR A LEBESGUE INTEGRAL

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In [1] De La Vallee' Poussin proved the following

THEOREM A. *Let ϕ be real-valued and absolutely continuous on the bounded closed interval $[t_1, t_2]$, and let M be a measurable subset of $[t_1, t_2]$ for which $m(\phi(M)) = 0$ where m is Lebesgue measure. Then $\phi'(X) = 0$ almost everywhere on M.*

This result was used by the same author in proving the following change of variable formula for a Lebesgue integral.

THEOREM B. *Let f be extended real-valued and summable over $[a, b]$. Let $F(X) = \int_a^X f(u)\,du$. Let ϕ be real-valued and absolutely continuous on $[t_1, t_2]$ such that $\phi([t_1, t_2]) \subset [a, b]$, and such that $F(\phi)$ is absolutely continuous on $[t_1, t_2]$. Let ϕ' be a function equal almost everywhere to the derivative of ϕ. Then $f(\phi)\phi'$ is summable over $[t_1, t_2]$ and*

$$\int_{\phi(t_1)}^{\phi(t_2)} f(u)\,du = \int_{t_1}^{t_2} f(\phi(t))\phi'(t)\,dt = F(\phi(t_2)) - F(\phi(t_1)).$$

Fichtenholz proved a version of change of variable formula in which ϕ is not required to be absolutely continuous, but is continuous monotone. See [2].

Nevertheless, as we shall show, the ϕ in the change of variable formula does not need to be either absolutely continuous or monotone. In fact, it need not even be continuous. Given all the other hypotheses of Theorem B, it suffices if ϕ is of bounded variation. The precise theorem we shall prove is stated as Theorem 2 below.

A key fact employed in the proof will be the following result which generalizes Theorem A and answers affirmatively some questions raised by Varberg in a recent paper [5].

THEOREM 1. *Let ϕ be real-valued and of bounded variation on the bounded closed interval $[t_1, t_2]$, and let M be a measurable subset of $[t_1, t_2]$, for which $m(\phi(M)) = 0$ where m is Lebesgue measure. Then $\phi'(X) = 0$ almost everywhere on M.*

THEOREM 2. *Let f be extended real-valued and summable over $[a, b]$. Let $F(X) = \int_a^X f(u)\,du$. Let ϕ be real-valued and of bounded variation on $[t_1, t_2]$ such that $\phi([t_1, t_2]) \subset [a, b]$; and such that $F(\phi)$ is absolutely continuous on $[t_1, t_2]$.*

Received August 14, 1967.